Optimal Auctions and Efficient Bargains

Optimal Auctions: A risk neutral seller has an object she values at v_0. There are n risk neutral buyers, 1,2,...,n, and buyer i has a value v_i known only to buyer i. *Ex ante*, the valuations are *i.i.d.* random variables drawn from the c.d.f. F, with continuous p.d.f. f, which has support [0,1,]. This is the *independent private values* environment.

A mechanism is a set of functions P_i and Y_i that map the vector of reported types $v=(v_1,...,v_n)$ into $\mathbf{R}$. P_i gives the probability the i gets the good, and Y_i gives i's expected payment. Since the seller only has one unit:

(1) $$P_i(v) \geq 0, \ \sum_{i=1}^{n} P_i(v) \leq 1.$$

We use the notation E_{-i} to be expectation over $(v_1,...,v_{i-1},v_{i+1},...,v_n)$, and let $y_i(v_i)=E_{-i}Y_i(v)$, $p_i(v_i)=E_{-i}P_i(v)$. Similarly, E_i is expectation over v_i, and $E = E_iE_{-i}$.

The buyer earns:

(2) $$\pi_i(v_i) = \max_r \ v_i p_i(r) - y_i(r)$$

Incentive compatibility, $\pi_i(v_i) = v_i p_i(v_i) - y_i(v_i)$, is equivalent to (see below):

(3) $$\pi_i'(v_i) = p_i(v_i), \text{ and}$$

(4) p_i nondecreasing.

We assume the buyers can choose whether to participate, which yields an individual rationality constraint $\pi_i \geq 0$; since π_i is nondecreasing by (3), this is equivalent to:

(IR) $$\pi_i(0) \geq 0.$$

(5) $$E_i y_i = \int_0^1 y_i(x) f(x)\, dx \qquad \text{(by def'n } E_i\text{)}$$

$$= \int_0^1 x p_i(x) f(x)\, dx - \int_0^1 \pi_i(x) f(x)\, dx \qquad \text{(by (2),IC)}$$

$$= \int_0^1 x p_i(x) f(x)\, dx + \pi_i(x)(1 - F(x))\Big|_0^1 - \int_0^1 \pi_i'(x)(1 - F(x))\, dx \qquad \text{(IBP)}$$

$$= -\pi_i(0) + \int_0^1 \left[x - \frac{1 - F(x)}{f(x)} \right] p_i(x) f(x)\, dx \qquad \text{(by (3))}$$

$$= -\pi_i(0) + E_i \left[\left[v_i - \frac{1 - F(v_i)}{f(v_i)} \right] p_i(v_i) \right] = -\pi_i(0) + E_i\, c(v_i) p_i(v_i), \qquad \text{(by def'n } E_i\text{)}$$

where $c(x) = x - \dfrac{1-F(x)}{f(x)}$.

Thus, the seller earns

$$
\begin{aligned}
S &= v_0 + \sum_{i=1}^{n} E_i[y_i(v_i) - v_0 p_i(v_i)] \\
&= v_0 + \sum_{i=1}^{n}\left[-\pi_i(0) + E_i(c(v_i) - v_0)p_i(v_i)\right] && \text{(by (5))} \\
&= v_0 + E\sum_{i=1}^{n}\left[-\pi_i(0) + (c(v_i) - v_0)P_i(v)\right] && \text{(def'n of } p_i\text{)}
\end{aligned}
$$

The seller wishes to maximize this subject to (1), (3), (4) and (IR). (IR) has the effect of setting $\pi_i(0)=0$. The strategy for maximizing S is then to ignore (4), maximize S subject to (1) and (3), and *hope* (4) is satisfied (i.e. find conditions on F so that (4) is satisfied). We obtain

$$
(6) \qquad P_i(v) = \begin{cases} 1 & \textit{if } c(v_i) \geq \max\limits_{j \neq i}\{v_0, c(v_j)\} \\ 0 & \textit{otherwise} \end{cases}
$$

We have found an optimum provided p_i is nondecreasing, which arises if c is nondecreasing.

Indirect Implementation: Any of the standard auctions (first price sealed bid, second price sealed bid (*Vickrey*), oral ascending (*English, Japanese*), oral descending (*Dutch*)) have the property that the highest value buyer obtains the good. Thus, by (5), they produce the same rents for the seller:

Revenue Equivalence Theorem: *Any standard auction with the same minimum payment produces the same profits for the seller.*

Let r satisfy $c(r)=v_0$. Then, provided c is nondecreasing, (6) reduces to selling the good to the highest value buyer, provided that buyer's value exceeds r. In addition, the mechanism must give no profits to the lowest type of buyer. Any mechanism with these properties will maximize the seller's profits.

Optimal Auctions Theorem: *If c is nondecreasing, any standard auction when supplemented with minimum payment or reserve price r maximizes the seller's profits over all possible selling mechanisms.*

IC Characterization: Let $\Pi(v,r)=vp(r)-y(r)$ be the profits of a buyer who reports r but has value v and $\pi(v) = \Pi(v,v)$. IC is given by $\Pi(r,v) \leq \Pi(v,v)$ for all r,v.

Lemma: IC iff $\pi'(v) = p(v)$ *a.e.* v and p is nondecreasing.

Proof: (→) The envelope theorem gives $\pi'(v) = p(v)$. Adding $\Pi(v,v) - \Pi(r,v) \geq 0$ to $\Pi(r,r) - \Pi(v,r) \geq 0$ gives $(v-r)(p(v)-p(r)) \geq 0$, which is equivalent to p nondecreasing.

(←)

$$
\begin{aligned}
\Pi(v,v) - \Pi(r,v) &= vp(v) - y(v) - vp(r) + y(r) \\
&= (r-v)p(r) + \pi(v) - \pi(r) \\
&= (r-v)p(r) + \int_r^v \pi'(\alpha)\, d\alpha \\
&= \int_r^v p(\alpha) - p(r)\, d\alpha \geq 0.
\end{aligned}
$$

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Myerson-Satterthwaite

Buyer has privately observed value b with *cdf* F, *pdf* f. Seller has privately observed value s with *cdf* G, *pdf* g. Both parties are risk neutral. The supports of the distributions are $[b_L, b_H]$ and $[s_L, s_H]$, respectively. To make the trading decision nontrivial, assume $b_L < s_H$ (worst types don't trade) and $b_H > s_L$ (best types trade).

The efficient allocation is defined by exchange iff $b > s$. Consider any mechanism trying to implement the efficient allocation. This mechanism charges the buyer $\beta(b)$ when the buyer reports b, and pays the seller $\sigma(s)$ when the seller reports s. Thus, the payoff to the buyer is $u(b) = \max_r bG(r) - \beta(r)$, and the payoff to the seller is $\pi(s) = \max_r \sigma(r) - s(1 - F(b))$. Incentive compatibility gives $u'(b) = G(b)$ and $\pi'(s) = -(1 - F(s))$. The individual rationality constraints are $u(b_L) \geq 0$ and $\pi(s_H) \geq 0$. Let $a \wedge b = \min\{a, b\}$. The buyer's payment is denoted B and the seller's earnings denotes S, so that the mechanism's average revenues are B-S.

$$B = \int_{b_L}^{b_H} \beta(x) f(x)\, dx = \int_{b_L}^{b_H} [xG(x) - u(x)] f(x)\, dx$$

$$= -[xG(x) - u(x)](1 - F(x)) \Big|_{b_L}^{b_H} + \int_{b_L}^{b_H} [xg(x) + G(x) - u'(x)](1 - F(x))\, dx$$

$$= b_L G(b_L) - u(b_L) + \int_{b_L}^{b_H} xg(x)(1 - F(x))\, dx = \int_{s_L \wedge b_L}^{s_H} xg(x)(1 - F(x))\, dx + \int_{s_L}^{b_L} G(x)\, dx.$$

$$S = \int_{s_L}^{s_H} \sigma(x) g(x)\, dx = \int_{s_L}^{s_H} [x(1 - F(x) + \pi(x)] g(x)\, dx$$

$$= \int_{s_L}^{s_H} x(1 - F(x)) g(x)\, dx + \pi(x) G(x) \Big|_{s_L}^{s_H} - \int_{s_L}^{s_H} \pi'(x) G(x)\, dx$$

$$= \int_{s_L}^{s_H} x(1 - F(x)) g(x)\, dx + \int_{s_L}^{s_H} (1 - F(x)) G(x)\, dx = \int_{b_L \wedge s_L}^{s_H} x(1 - F(x)) g(x)\, dx + \int_{s_L}^{s_H} (1 - F(x)) G(x)\, dx.$$

$$B - S = -\int_{\max\{b_L, s_L\}}^{s_H} (1 - F(x)) G(x)\, dx < 0.$$

Myerson-Satterthwaite Theorem: *Any mechanism implementing efficient exchange in the bilateral trading problem loses money on average.*

Remark: Because of discreteness of the good, the M-S environment doesn't equate the marginal value of the buyer to the marginal value of the seller. When the good is made divisible, efficient exchange becomes possible for a large class of environments; see McAfee, "Efficient Allocation with Continuous Quantities," *JET*, 1991. In addition, the result is sensitive to small amounts of correlation in s and b: see McAfee and Reny, *Econometrica*, 1992, which also shows that correlation permits the seller in an optimal auction to extract all the surplus.

General IC Characterization: The IC characterization is a special case of a much more general theorem, first proved by Guesnerie and Laffont (*J Pub E*, 1984). Suppose there are two goods x,y and an agent of type $t \in [0,1]$ has utility $u(x,y,t)$ of consuming (x,y). u is increasing in x and y. Define the shadow price $p(x,y,t)$, the marginal rate of substitution, by $p(x,y,t) = \dfrac{u_x(x,y,t)}{u_y(x,y,t)}$. Subscripts denote partial derivatives. The single crossing property is given by

(SCP) $\quad p_t(x,y,t) > 0.$

Note that SCP holds automatically for the linear type problems so far considered (y is the negative of payments, x the probability of receiving the good, t the value of the good). A direct, deterministic mechanism maps reported types into $\mathbf{R}^2$, providing the consumption $(x(r),y(r))$ to an agent that reports type r. Incentive compatibility is,

(IC) $\quad u(x(t),y(t),t) \geq u(x(r),y(r),t).$

Theorem: *Suppose SCP holds. Then (IC) if and only if*

(*) $\quad \dfrac{d}{dt}u(x(t),y(t),t) = u_t(x(t),y(t),t),$

(**) $\quad$ *x nondecreasing.*

Sketch of Proof: SCP insures that utility isoquants in (x,y) space only cross once, and moreover, the higher type's isoquant has steeper slope than a lower type's isoquant.
($\rightarrow$) (*) is the envelope theorem. Let $s<t$. In figure 1, the dark curves represent indifference curves for types t and s, both through $(x(s),y(s))$. The vertical lines shade the region where $(x(t),y(t))$ must lie for s's incentive constraint. The horizontal lines shade the region for t's incentive constraint. Thus IC implies $(x(t),y(t))$ lies in the cross-hatched region, proving (**), and in addition, that y is nondecreasing.
($\leftarrow$) Consider the graph of $(x(t),y(t))$. Note that (*) implies, at points t of differentiability of x,y,

(***) $\quad \dfrac{y'(t)}{x'(t)} = p(x(t),y(t),t).$

At points of discontinuity in (x,y) $\lim_{r \to t} u(x(r),y(r),t) = u(x(t),y(t),t)$, otherwise (*) must fail. Thus, we can "complete" the graph, connecting discontinuities in such a way that $u(x,y,t)$ is constant, and therefore (***) holds on this connected graph. Thus, since x is nondecreasing (and (*) then implies y is nonincreasing), at almost every point on the graph, (***) holds. Fix t, and consider Figure 2. To the left of $(x(t),y(t))$, the isoquant for type t cuts the (x,y) downward, by SCP, [and conversely to the right of this point], since the graph of (x,y) is tangent to the isoquant for a lower [higher] type. But this implies (IC). ■