Before going to Europe on business, a man drove his Rolls-Royce to a downtown NY City bank and went in to ask for an immediate loan of $5,000. The loan officer, taken aback, requested collateral. "Well, then, here are the keys to my Rolls-Royce", the man said. The loan officer promptly had the car driven into the bank's underground parking for safe keeping, and gave him $5,000.

Two weeks later, the man walked through the bank's doors, and asked to settle up his loan and get his car back. "That will be $5,000 in principal, and $15.40 in interest", the loan officer said. The man wrote out a check and started to walk away.

"Wait sir", the loan officer said, "while you were gone, I found out you are a millionaire. Why in the world would you need to borrow $5,000?"

The man smiled. "Where else could I park my Rolls-Royce in Manhattan for two weeks and pay only $15.40?"

Investment

Net Present Value

- Stream of payments A_0, A_1, …

$$NPV = A_0 + \frac{A_1}{1+r_1} + \frac{A_2}{(1+r_1)(1+r_2)} + \frac{A_3}{(1+r_1)(1+r_2)(1+r_3)} + \ldots$$

- Consol: same payment forever
- Common interest rate r

$$v = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots = \frac{1}{1-\frac{1}{1+r}} - 1 = \frac{1}{r}.$$

Mortgage

- Pay \$1 per period, *n* periods

$$NPV = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + ... + \frac{1}{(1+r)^n}$$

$$= \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Car Loan

- *r* = ½% per month
- \$1/ month for 60 months =

$$NPV = \frac{1}{.005}\left(1 - \frac{1}{(1.005)^{60}}\right) = 51.73$$

- To borrow \$20,000 requires

$$\frac{\$20,000}{51.73} = \$386.66 / mo.$$

Value of the Lottery

- \$1M/yr, 20 years

$$PV = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + ... + \frac{1}{(1+r)^{19}} = 1 + \frac{1}{r}\left(1 - \frac{1}{(1+r)^{19}}\right).$$

r	3%	4%	5%	6%	7%	10%
PV (000s)	\$15,324	\$14,134	\$13,085	\$12,158	\$11,336	\$9,365

Bond Price

- Bond pays $1000 in 10 years

$$\text{Price} = NPV = \frac{\$1000}{(1+r)^{10}}$$

- Interest rate increase causes bond prices to fall

MBA Investment Strategy

- Compute NPV
- Undertake project if NPV > 0
- Preferable to calculating internal rates of return (solve equation NPV=0 for *r*) because IRR not well defined

Investment Under Uncertainty

MBA Strategy:

- Use risk-adjusted interest rate
- Risk adjustment for project, not parent!
  - Ibbotson Cost of Capital Yearbook
- Compute expected NPV
- Undertake if E NPV > 0

Rates of Return

- If you had invested $1 in the following from end of 1925 to end of 1999 it would have increased to

Asset Class	Annual Return	Ending Wealth
S&P 500	11.3%	$2,845.63
Small company stock index	12.6%	$6,640.79
Long-term corporate bond index	5.6%	$56.38
Long-term government bond index	5.1%	$40.22
Intermediate-term government bond index	5.2%	$43.93
U.S. Treasury Bills	3.8%	$15.64
Inflation	3.1%	$9.39

Risk and Return

Asset Class	Geometric Mean	Standard Deviation	Arithmetic Mean
Small company stocks	12.6%	33.6%	17.6%
Large company stocks	11.3%	20.1%	13.3%
Long-term corporate bonds	5.6%	8.7%	5.9%
Long-term government bonds	5.1%	9.3%	5.5%
Intermediate-term gov't bonds	5.2%	5.8%	5.4%
U.S. Treasury Bills	3.8%	3.2%	3.8%
Inflation	3.1%	4.5%	3.2%

Mean Returns

- Geometric mean $\left(\prod_{i=1}^{n} a_i\right)^{1/n}$
- Arithmetic mean $\frac{1}{n}\sum_{i=1}^{n} a_i$

$$Log\left[\left(\prod_{i=1}^{n} a_i\right)^{1/n}\right] = \frac{1}{n}\sum_{i=1}^{n} Log[a_i] \le Log\left[\frac{1}{n}\sum_{i=1}^{n} a_i\right]$$

Mean Returns, Continued

- Geometric makes sense when using rates of return over several years
- Arithmetic would be used for expected returns in a given year

Option Value

- NPV is appropriate for "*now or never*" decisions
- *Now or later* requires an additional consideration
  - Sell a painting
  - Drill an oil well
  - Build a factory
- Investment destroys option to invest

Example

- Spend $C < 1$ to produce a value V
- $V \sim U[0,1]$, interest rate r
- Use cutoff value V_0: invest if $V \geq V_0$
- Produces NPV = $J(V_0)$

$$J(V_0) = (1 - V_0)\left(\frac{1+V_0}{2} - C\right) + V_0\left(\frac{1}{1+r} J(V_0)\right).$$

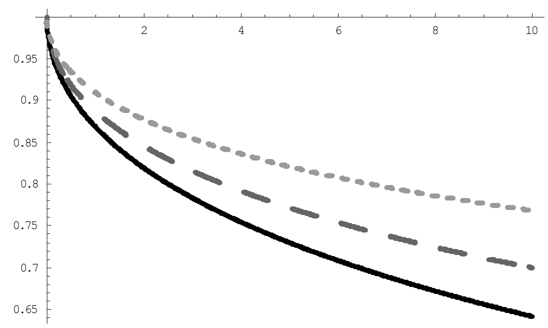
Investment Value

$$J(V_0) = \frac{(1 - V_0)\left(\frac{1 + V_0}{2} - C\right)}{1 - \frac{V_0}{1 + r}}.$$

- Maximized when

$$V_0 = (1 + r) - \sqrt{r^2 + 2r(1 - C)}.$$

V_0 for C = 0, 0.25, 0.5



Resource Extraction

- Fixed supply of a resource R
- Constant demand elasticity ε
- Let Q_t represent the quantity consumed at time t.
- Arbitrage: price rises at interest rate

$$aQ_0^{-1/\varepsilon}(1 + r)^t = p(Q_0)(1 + r)^t = p(Q_t) = aQ_t^{-1/\varepsilon}$$

Resource Use

$$R = (Q_0 + Q_1 + Q_2 + \ldots)$$

$$= Q_0\left(1 + (1+r)^{-\varepsilon} + (1+r)^{-2\varepsilon} + \ldots\right) = \frac{Q_0}{1-(1+r)^{-\varepsilon}}$$

- Arbitrage spreads use out
  - Never run out
- Prices rise at interest rate
- Markets don't view natural resources this way
- reflecting alternatives, technological change
- $r = 0.05$, $\varepsilon=2$, $1 - (1+r)^{-\varepsilon} = 9.3\%$ annual
  - *Half life 7 years*

Tree-Cutting

- Time to harvest then replant
  - Trees, lobsters, fish, cows
- Value after growth *t* of *b*(*t*)
- Continuous time interest rate

$$e^{-\delta t} = \left(\frac{1}{1+r}\right)^t$$

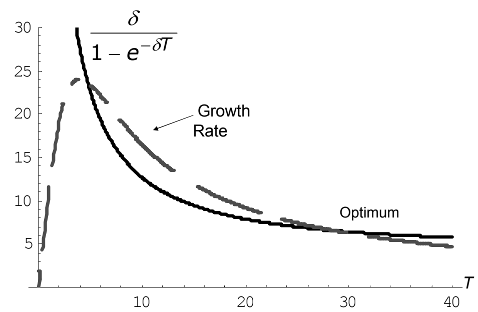
NPV

$$e^{-\delta T}pb(T) + e^{-2\delta T}pb(T) + e^{-3\delta T}pb(T) + \ldots$$

$$= \frac{e^{-\delta T}pb(T)}{1-e^{-\delta T}} = \frac{pb(T)}{e^{\delta T}-1}$$

- FOC $\quad \dfrac{b'(T)}{b(T)} = \dfrac{\delta}{1-e^{-\delta T}}$

Solution



Tree-Cutting

- Ramsey Rule: cut down the trees when they are growing at the interest rate

$$\frac{b'(T)}{b(T)} = \frac{\delta}{1-e^{-\delta T}}$$

- Approximately correct
- US policy of maximum sustainable yield sends $\delta \to 0$, yields $\frac{b'(T)}{b(T)} = \frac{1}{T}$

Collectibles

- time t starts 0 to ∞
- Quantity supplied $q(t) = q_0 e^{-\delta t}$
- ε = elasticity of demand
- g = growth rate of population
- r = discount rate (e^{-rt})

Porsche Speedster

Demand and Supply

- Demand $x_d(p,t) = ae^{gt}p^{-\varepsilon}$
- Demand and supply equate to give the marginal use value of an owner at time *t*

$$q_0 e^{-\delta t} = q(t) = x_d(v,t) = ae^{gt}v^{-\varepsilon}$$

or

$$v = \left(\frac{a}{q_0}\right)^{1/\varepsilon} e^{\frac{\delta+g}{\varepsilon}t}$$

Determination of Price

- Marginal holder must be just indifferent to holding
- Marginal holder who buys at *t* and sells at *t*+Δ gets

$$\int_0^{\Delta} e^{-ru}(v-s)du - p(t) + e^{-r\Delta}e^{-\delta\Delta}p(t+\Delta)$$

Per Period Value of Holding

$$\lim_{\Delta \to 0} \frac{1}{\Delta}\int_0^{\Delta} e^{-ru}(v-s)du - \frac{p(t)}{\Delta} + \frac{e^{-(r+\delta)\Delta}p(t+\Delta)}{\Delta}$$

$$= \lim_{\Delta \to 0} v - s + \frac{p(t+\Delta)-p(t)}{\Delta} - \frac{1-e^{-(r+\delta)\Delta}}{\Delta}p(t+\Delta)$$

$$= v - s + p'(t) - (r+\delta)p(t)$$

Marginal Owner is Indifferent

- Marginal owner $v = \left(\frac{a}{q_0}\right)^{1/\varepsilon} e^{\frac{\delta+g}{\varepsilon}t}$
- Yields

$$p'(t) = (r+\delta)p(t) + s - v = (r+\delta)p(t) + s - \left(\frac{a}{q_0}\right)^{1/\varepsilon} e^{\frac{\delta+g}{\varepsilon}t}$$

Solution

$$p(t) = e^{(r+\delta)t}\left(p(0) + \frac{1-e^{-(r+\delta)t}}{(r+\delta)}s - \left(\frac{a}{q_0}\right)^{1/\varepsilon}\frac{1-e^{-\left(r+\delta-\frac{\delta+g}{\varepsilon}\right)t}}{r+\delta-\frac{\delta+g}{\varepsilon}}\right)$$

Necessary Conditions

- Present value of marginal use value is finite: $r+\delta-\frac{\delta+g}{\varepsilon}>0$
- Not everyone wants to own the good:

$$\lim_{t\to\infty} e^{-rt}p(t)<\infty.$$

Starting Price

- Either $p(0)=\left(\frac{a}{q_0}\right)^{1/\varepsilon}\frac{1}{r+\delta-\frac{\delta+g}{\varepsilon}}-\frac{1}{(r+\delta)}s\geq 0$

or $0=p(0)=\left(\frac{a}{q(0)}\right)^{1/\varepsilon}\frac{1}{r+\delta-\frac{\delta+g}{\varepsilon}}-\frac{1}{(r+\delta)}s$

and $q(0)\leq q_0$

Implications

- May destroy some quantity initially
- Price rises exponentially
- Storage costs enter linearly

$$p(t)=\left(\frac{a}{q(0)}\right)^{1/\varepsilon}\frac{e^{\frac{\delta+g}{\varepsilon}t}}{r+\delta-\frac{\delta+g}{\varepsilon}}-\frac{s}{r+\delta}$$