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STABILITY OF EQUILIBRIA WITH EXTERNALITIES*

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It is shown that, in a class of models with multiple externalities (one positive and one negative), all stationary equilibria may be locally stable to perturbations, in the sense that there exist perfect foresight trajectories leading back to the equilibrium. Thus, scale diseconomies (arising, for example, out of a common resource pool) generally overturn the Liviatan-Samuelson result, that equilibria are either saddlepoints or sources.

Recent contributions by Diamond [1982], Howitt [1985], Howitt and McAfee [1987], and others have shown how externalities in the transactions technology of an economy can generate various Keynesian phenomena without any of the price rigidities typical of Keynesian analysis. In particular, these papers show how there can exist multiple stationary equilibria, with different levels of unemployment. The low level equilibria are in many respects similar to the persistent states of unemployment depicted by Keynes. The present paper addresses the question of the dynamic stability of these equilibria.

In well-behaved one-dimensional dynamic systems with multiple equilibria, successive equilibria alternate between locally stable and locally unstable. The analogue to this result in well-behaved dynamic optimization models with one state variable is the result of Liviatan and Samuelson [1969] that the two-dimensional system describing the motion of the system has equilibria that alternate (as the equilibrium value of the state variable changes) between saddlepoint stable and unstable. If one can argue that there is a unique equilibrium trajectory starting from any initial value of the state variable, then the saddlepoints are locally stable in the economic sense because the second dimension of the dynamic system is a

*This paper is a revision of our earlier discussion paper [1984]. We are grateful to Paul Romer for suggesting a reformulation of the analysis of that paper.

costate variable, or a rate of change of the state variable, whose initial value is not given by history but can adjust to put the economy on the unique stable trajectory passing through the given initial value of the state variable.

If this usual result were true in trade-externality models, then in the simplest case (e.g. Figure I below) of two nondegenerate equilibria only one would be observable under occasional perturbations, the other being unstable. Furthermore, intuition suggests that the low level equilibrium would be the unstable one. If true, this would obviously reduce the empirical importance of the multiplicity result. The result could perhaps be used to explain the complete shutting down of the economy because the origin in Figure I may be a (degenerate) saddlepoint equilibrium. But to explain an observable low level equilibrium short of this catastrophe (in the economic sense), one would have to rule out the simple configuration of Figure I.

Recent papers on price-level dynamics (e.g. Calvo, [1979]) show that the stability properties of dynamic optimization models do not carry over to perfect-foresight equilibrium models. These papers have shown how the equilibrium trajectories in a perfect-foresight model of an economy with a single-state variable can be characterized by a two-dimensional dynamic system that bears a

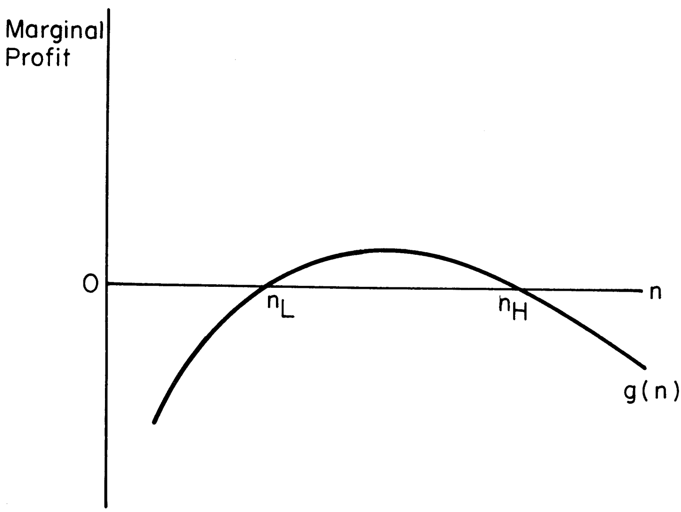

FIGURE I
Multiple Equilibria

superficial resemblance to the Euler equation of an analogous optimization model. But they exhibit stationary equilibria that can be locally stable under this system, a phenomenon ruled out in optimization models by the Liviatan-Samuelson result.

The present paper examines a perfect-foresight equilibrium model of an economy with trade externalities and with a single state variable. The main result is that every other stationary equilibrium is a saddlepoint, as in the Liviatan-Samuelson result, but the nonsaddle stationary equilibria can be locally stable, as in the Calvo model. In the case illustrated by Figure I, the high-level equilibrium will be the saddlepoint. Thus, the low level equilibrium of Figure I may be locally stable, and hence may be observable under perturbations.

More specifically, we show that one can get local stability of the nonsaddle equilibria if, in addition to the external economy emphasized by Diamond and others, there is also a diseconomy of scale, according to which the marginal adjustment cost faced by a firm trying to expand its activity level at a given rate is positively related to the activity level already attained by its rivals. An example of this diseconomy is the effect analyzed in our earlier paper [1987], whereby an increase in aggregate employment raises a firm's recruiting cost by reducing the rate at which unemployed job searchers contact the firm. Furthermore, we show that this diseconomy is necessary for our result because without it the model's trajectories can be characterized as solutions to the Euler equation of a social planning problem, to which the Liviatan-Samuelson local result applies.[1]

As in the price-level-dynamics literature, this local stability result implies an indeterminacy of equilibrium when the economy begins in the neighborhood of a low level equilibrium. The penultimate section of the paper discusses the implications of the indeterminacy and points out how it can apply even in the neighborhood of the high level equilibrium.

Most of the paper is motivated by the macroeconomic questions addressed by the trade-externality literature. But the abstract formal model that we use admits of several microeconomic interpretations, which we discuss in the final section of the paper.

1. Diamond and Fudenberg (1987) also show the possibility that the lower of two nondegenerate stationary equilibria can be a sink in the Diamond [1982] model of trade externalities. Their analysis, which is directed toward showing the existence of deterministic cycles, does not show the dependence of this result on an expansion-diseconomy.

I. The Model

We postulate a model with a large number of identical firms, and an even larger number of identical households, all risk neutral and all with the same constant rate of pure time preference. There are three tradable objects: output, homogeneous labor services, and money. The money is a pure accounting device. A firm's receipts are instantaneously transferred to its workers and owners and must be used for purchasing output from other firms during the current period. No credit market is assumed to exist, but that is no restriction in a world of risk neutrality and identical rates of time-preference.

The output market is perfectly competitive, in that all firms and households perceive a perfectly elastic demand schedule, and aggregate demand always equal aggregate output. But, as in the analyses of Hahn [1971], Niehans [1971], and others, firms must incur a transaction cost to operate in the market. This transaction cost takes the form of output used up in the selling process. Thus, a firm employing n units of labor will have a gross revenue of $f(n)$, where f is its production function, and will pay a total transaction cost of $\sigma(n,\bar{n})$, where $\bar{n}$ is aggregate employment (per firm).

This transaction cost depends upon the firm's own employment because the more it sells the greater the required cost. It depends upon aggregate employment (per firm) $\bar{n}$ because of the trade externality. The larger is $\bar{n}$, the greater is the equilibrium level of aggregate demand, and by assumption, the less the cost of selling a given quantity. This effect may be rationalized in a number of different ways. For example, as aggregate demand goes up, the rate of arrival of buyers to a store may increase, thereby reducing the costs of advertising, or the size of the average customer's purchase may go up, thereby allowing the same quantity to be sold with fewer individual sales. Rather than be specific about the source of the externality, we shall merely take the function σ as given.

The typical firm's wage bill will be $w(n,\bar{n})$. This function can be derived in a number of different ways: for example, if the labor market is perfectly competitive when $w(n,\bar{n}) = w^s(\bar{n}) \cdot n$, where w^s is the supply price of $\bar{n}$ units of labor (per firm). On the other hand, if the labor market is a search market as in Diamond [1982] or Howitt and McAfee [1987], then $w(n,\bar{n})$ will represent the predictable outcome of a bargaining process that takes place between a firm and n workers (each inelastically supplying one unit) when aggregate employment is $\bar{n}$.

The firm also faces costs of hiring—costs which it incurs in the form of output used up. The costs are given by the function $\gamma(\dot{n} + \delta n, \bar{n})$, where δ can be interpreted as either the death rate of workers or the exogenous rate at which job separations occur for noneconomic reasons. The expansion-cost function γ is written as depending upon $\dot{n} + \delta n$ because under the assumption that employed workers do not search for a job elsewhere (which makes sense in a symmetric equilibrium with homogeneous firms and workers), this will be the firm's gross rate of hiring. It is written as depending upon $\bar{n}$ to allow for the diseconomy referred to earlier. This diseconomy can be thought of as arising from the common property nature of the pool of unemployed job searchers. As $\bar{n}$ increases, the size of the pool from which the typical firm draws its new recruits is thereby reduced.[2] This makes it more difficult for the firm to find any given number of new recruits. Thus, if this externality is present, $\gamma_2 > 0$, and $\gamma_{12} > 0$. We shall also be interested in the limiting case where the externality is absent, defined by the conditions $\gamma_2 = \gamma_{12} = 0$.

The flow of instantaneous profits to a firm is given by the function,

$$L(\dot{n},n,\bar{n}) \equiv \Pi(n,\bar{n}) - \gamma(\dot{n} + \delta n, \bar{n}) \equiv f(n) - \sigma(n,\bar{n}) - w(n,\bar{n}) - \gamma(\dot{n} + \delta n, \bar{n}), \tag{1}$$

where $\Pi(n,\bar{n})$ is its gross profit function (gross of expansion costs). Assume that

(2) L is almost everywhere twice continuously differentiable, and for any $\bar{n}$ it is concave in $(\dot{n},n)$, with $L_{11} = -\gamma_{11} < 0$.

The properties of stationary equilibria depend upon the function,

$$g(n) \equiv L_2(0,n,n) + rL_1(0,n,n) \equiv \Pi_1(n,n) - (r + \delta)\gamma_1(\delta n,n) \equiv f'(n) - \sigma_1(n,n) - w_1(n,n) - (r + \delta)\gamma_1(\delta n,n), \tag{3}$$

where r is the rate of time preference. This function describes the steady state marginal profit of employment; i.e., the marginal contribution of a unit of labor to the stationary flow of profits: $L_2 = \Pi_1 - \delta\gamma_1$, minus the interest cost of the initial recruiting outlay—

2. We are assuming here that if there is an "encouraged worker" effect on participation rates, a unit increase in $\bar{n}$ raises the labor force by less than one unit. For a more detailed discussion of this diseconomy, see Section IV below.

$-rL_1 = r\gamma_1$. Note that g describes only the private component of this marginal profit. We make two assumptions on g:

(4) $$g(n) < 0 \text{ for all } n \text{ in some interval } (0,n_1);$$

(5) $$g(n) < 0 \text{ for all } n > n_2, \text{ where } n_2 > n_1.$$

Assumption (4) is a generalization of the notion in the models of Diamond [1982] and Howitt [1985], that, because of the trade externality, if the entire economy is operating at too low a level, then the transaction cost of selling even a single unit outweighs the benefit. It would follow under normal assumptions if, for example, $\sigma(n,\bar{n})$ were proportional to $f(n)$ but inversely proportional to the level of aggregate demand $f(\bar{n})$, as in the example (15) below. Assumption (4) is the only place in the formal analysis where the external economy in trading is invoked.

Assumption (5) asserts that eventually some combination of the rising cost of expansion, decreasing returns, and the finiteness of the economy's endowment of labor overcome the external trade economy and reduce the marginal profit below zero. It could be derived under very general assumptions on f, σ, and w if, for example, the cost of hiring approached infinity as $\bar{n}$ approached the economy's endowment of labor. But it is important to note that the external diseconomy of expansion which this would require is not necessary for any of the above assumptions. Thus, (5) could be derived by assuming a competitive labor market in which the supply price of labor goes to infinity as employment approaches the economy's total endowment, or by assuming an infinitely elastic supply of labor but a marginal expansion cost that goes to infinity with the gross rate of hiring. We shall provide examples in Section IV, both with and without the external diseconomy. The latter illustrates the importance of the external diseconomy in achieving stability of all stationary equilibria.

II. Equilibria

We analyze symmetric perfect foresight equilibrium trajectories for the economy. Each firm can foresee perfectly the time path of aggregate employment $\{\bar{n}(\tau)\}_{\tau=0}^{\tau=\infty}$. Given this path and an initial employment-level $n(0)$, the firm chooses the time path of its own employment so as to

(6) $$\max \int_0^\infty e^{-rt} L(\dot{n},n,\bar{n})\, dt.$$

It follows from well-known results that, given (2), a path $n(t)$ solves the firm's decision problem only if it satisfies the Euler equation:

$$L_2(\dot{n},n,\bar{n}) + rL_1(\dot{n},n,\bar{n}) = \frac{d}{dt} L_1(\dot{n},n,\bar{n}), \tag{7}$$

and that it provides a solution to the problem if it satisfies the Euler equation and the transversality condition:

$$\lim_{t\to\infty} e^{-rt}L_1(\dot{n},n,\bar{n}) = 0. \tag{8}$$

An equilibrium trajectory for the economy is defined as a piece-wise differentiable time path for employment such that if each firm takes it as the given path of aggregate employment, then it will also choose it as the path of its own employment. Thus, along an equilibrium trajectory the Euler equation must be satisfied with $n(t) \equiv \bar{n}(t)$; that is, for $n(t)$ to be an equilibrium trajectory, it must solve the second-order differential equation:

$$L_2(\dot{n},n,n) + rL_1(\dot{n},n,n) = \frac{d}{dt} L_1(\dot{n},n,n). \tag{9}$$

Furthermore, any solution to the "Euler" equation (9) is necessarily an equilibrium trajectory if it also satisfies the "transversality" condition:

$$\lim_{t\to\infty} e^{-rt}L_1(\dot{n},n,n) = 0. \tag{10}$$

The stationary equilibria of the model are equilibrium trajectories with constant employment. They correspond to the rest points of the system (9) because any rest point obviously satisfies (10). They are thus defined by (9) together with the condition $\dot{n} = \ddot{n} = 0$. It follows immediately that at any rest point n^*, the marginal profit of employment just equals zero:

$$g(n^*) = L_2(0,n^*,n^*) + rL_1(0,n^*,n^*) = 0. \tag{11}$$

It follows also that any n^* satisfying (11) is a rest point because when $n = n^*$ and $\dot{n} = 0$, then according to (9), $\ddot{n} = L_{11}^{-1} \cdot g(n^*) = 0$. Thus, stationary equilibria correspond to the solutions to (11).

Assumptions (4) and (5) guarantee that if there is a nonzero stationary equilibrium, it must occur between n_1 and n_2, and if there exists one such equilibrium, then there will exist at least one other, except in the razor's edge case where $0 = \max g(n)$. The simplest case of two nonzero stationary equilibria is shown in Figure I.

III. Local Stability and Observability

Except in razor's edge cases that we ignore, the local stability properties of a stationary equilibrium n^* under the system (9) are determined by the linear approximation to (9) (a complete derivation is given in the Appendix):

$$(12)\qquad 0 = L_{11}\ddot{n} + (L_{13} - rL_{11})\dot{n} - (L_{22} + L_{23} + rL_{12} + rL_{13}) \times (n - n^*)$$
$$= -\gamma_{11}\ddot{n} + (r\gamma_{11} - \gamma_{12})\dot{n} - g'(n^*)(n - n^*),$$

where all partial derivatives are evaluated at $(0,n^*,n^*)$. The roots of this system, (λ_1,λ_2) must satisfy

$$(13)\qquad \lambda_1 + \lambda_2 = r - (\gamma_{12}/\gamma_{11}),$$

$$(14)\qquad \lambda_1\lambda_2 = g'(n^*)/\gamma_{11}.$$

Saddle-stability will occur if there are two real roots: one positive, and the other negative. This is equivalent to the condition $\lambda_1 \cdot \lambda_2 < 0$. By (14) and (2) this is equivalent to the condition, $g'(n^*) < 0$. Since $g(n^*) = 0$, our first result is that every other stationary equilibrium, i.e., each one at which the curve $g(n)$ cuts the horizontal axis from above, will be a saddlepoint, and that the intervening equilibria, at which $g'(n^*) > 0$, will not be saddlepoints. Furthermore, the assumptions (4) and (5) guarantee that the lowest level equilibrium will be one at which $g'(n^*) > 0$ and the highest level equilibrium will have $g'(n^*) < 0$. Thus, the lowest level equilibrium will not exhibit saddle-stability but the highest level equilibrium will.

According to (13), the nonsaddle equilibria will be locally stable or unstable, depending upon the sign of $r - \gamma_{12}/\gamma_{11}$. From this follows our next result, that if there is no external diseconomy in expansion—i.e., $\gamma_{12} = 0$—then the nonsaddle equilibria must all be locally unstable, because $\lambda_1 + \lambda_2 = r > 0$. In this case, the "usual" result holds, and the low level equilibrium of Figure I is unobservable under perturbations. There is no equilibrium trajectory starting anywhere other than at the low level equilibrium itself that converges to it.

If (9) were the Euler equation to an optimization model with concave maximand, our result in the absence of expansion-diseconomies would be a direct consequence of the Liviatan-Samuelson result. This is almost the case, because (9) is indeed the Euler equation for the problem of maximizing

$$\int_0^\infty e^{-rt}\{H(n) - \gamma(\dot{n} + \delta n)\}\,dt, \text{ where } H(n) \equiv \int_0^n \Pi_1(x,x)\,dx.$$

But in the neighborhood of any nonsaddle equilibrium, where $g'(n^*) > 0$, the integrand of this problem is not concave, since at the equilibrium,

$$\begin{aligned}\frac{\partial^2}{\partial n^2}\{H(n) - \gamma(\dot{n} + \delta n)\} &= \frac{d}{dn}\{\Pi_1(n,n) - \delta\gamma_1(\delta n)\}\\ &= \frac{d}{dn}\{g(n) + r\gamma_1(\delta n)\}\\ &> g'(n)\\ &> 0.\end{aligned}$$

The main result of the paper is that in the presence of the expansion-diseconomy the nonsaddle equilibria like n_L of Figure I may be locally stable. By our previous results this will happen if at these equilibria, $\gamma_{12}/\gamma_{11} > r$. Obviously, this will hold in the limiting case of no discounting. By continuity it will also happen with a small enough rate of discount. An example with positive discounting is provided in the following section.

The local stability of n_L means that it is observable under occasional perturbations. For any n in a neighborhood of n_L, there will be a continuum of solutions of the "Euler" equation (9) that converge upon n_L, one for each initial value of $\dot{n}$ in the neighborhood of zero. Any such convergent solution will obviously satisfy the "transversality" condition (10) and will thus be an equilibrium trajectory. Thus, even if n is displaced a little from n_L, there will be equilibrium trajectories that return asymptotically to n_L.

More generally, any stationary equilibrium that is either a saddlepoint or has $\lambda_1 + \lambda_2 < 0$ is observable under perturbations. A saddlepoint is observable because starting at any initial n in its neighborhood there will be a unique solution to (9) converging monotonically to it. Thus, contrary to the "usual" result, it is possible for all stationary equilibria to be observable under perturbations.

It is useful to contrast this outcome, the local stability of stationary equilibria, with the more conventional models featuring multiple equilibria. In these models a slight increase in activity at a low level equilibrium makes marginal profit positive, encouraging further expansion toward the high level equilibrium.

By contrast, in our model the value of expansion depends on future levels of activity of the other agents. Thus, if the activity level n is increased beyond the low level equilibrium n_L, this increases only the instantaneous gains to expansion. However, since current expansion produces a stream of returns, the marginal value of n increases only if the future value of activity will be high enough, which depends on the discount rate r. Effectively, if a firm believes that the other firms will not expand farther, that firm will not be induced to undertake further expansion. Around the low level equilibrium, there are generally at least two sets of perfect foresight beliefs about other firms' activity levels: one in which all firms expand to a high level of activity, and one in which the activity level returns to the low level. Believing the latter to be true drastically reduces the gains to expansion, even when returns are (instantaneously) higher than at n_L which supports the stability of the low level equilibrium.

IV. Examples

An example of the economy satisfying (2), (4), and (5) with an external diseconomy of expansion is

$$(15)\quad \begin{cases} f(n) = f \cdot n, \quad f > 0; \quad \sigma(n,\bar{n}) = n \cdot \min(f, \sigma/\bar{n}), \quad \sigma > 0; \\ w(n,\bar{n}) = n\left(\dfrac{\bar{n}}{\xi - \bar{n}}\right), \quad \xi > \dfrac{\sigma}{f}, \\ \gamma(\dot{n} + \delta n, \bar{n}) = \left(\dfrac{\dot{n} + \delta n}{\alpha(\xi - \bar{n})}\right)^2, \quad \alpha > 0 \\ L \text{ defined on } RX(0,\xi)^2. \end{cases}$$

In this example ξ represents the economy's endowment of labor. As aggregate employment approaches ξ, the example assumes that both the wage and the marginal cost of hiring at any given rate go to infinity.

Note that, from (3) and (15),

$$(16)\quad g(n) = f - \min(f,\sigma/n) - n/(\xi - n) - 2(r + \delta)\delta n/\alpha^2(\xi - n)^2.$$

Thus, $g(n) < 0$ for $n < \sigma/f$, implying that the example satisfies (4), and $\lim_{n\to\xi} g(n) = -\infty$, implying that it also satisfies (5).

The expansion-cost function of (15) might be derived as follows. Suppose, as in our earlier paper [1987] that for a firm to hire workers, it must first attract them into their "recruiting net," and that the rate at which searching workers enter a net of unit size is proportional to α, their speed of search, and U, the number of unemployed searchers. Assume a zero value of leisure. Then $U = \xi - \bar{n}$, where ξ is the constant, exogenous size of the labor force (all employment magnitudes are per firm). Assume that the speed of search is a constant. Assume that the firm has no way of influencing the search activity of workers that have not yet fallen into its net and that the rate at which it attracts workers into the net is proportional to the size of the net θ. Then the firm's gross rate of hiring will equal the rate at which searching workers enter the firm's net, which can be written as (by normalizing):

$$\dot{n} + \delta n = \theta \cdot \alpha(\xi - \bar{n}) \tag{17}$$

because it would be privately inefficient for each "contact" not to result in a job match. Assume that the firm may vary its net size according to the quadratic cost function, $\gamma = \theta^2$; then the expansion cost function in (15) follows directly from (17).

An example of an economy satisfying (2), (4), and (5) without the external diseconomy is constructed by replacing the expansion-cost function in (15) with the pure quadratic function, $\gamma = (\dot{n} + \delta n)^2$.

To construct an example in which there are exactly two nondegenerate stationary equilibria, as in Figure I, with the two-level equilibrium being locally stable under the "Euler" equation, take the special case of (15) with $\alpha = \delta = \xi = \sigma = 1, f = 8$, and $r = \frac{1}{5}$. From (16),

$$g(n) = \begin{cases} 8 - \dfrac{1}{n} - \dfrac{n}{1-n} - \dfrac{2(\frac{6}{5})n}{(1-n)^2}; & \dfrac{1}{8} < n < 1 \\[2ex] -\dfrac{n}{1-n} - \dfrac{2(\frac{6}{5})n}{(1-n)^2}; & 0 < n \leqq \dfrac{1}{8}. \end{cases} \tag{18}$$

A stationary equilibrium is an n in the interval (0,1) with $g(n) = 0$. By (18) such an n must lie in the subinterval $(\frac{1}{8},1)$ because $g(n) < 0$ on $(0,\frac{1}{8})$. But within that subinterval it is easily verified that g is strictly concave, that $g < 0$ at each end of the subinterval, and that g attains positive values inside the subinterval (because $g(\frac{1}{2}) = \frac{1}{5}$). Hence there are exactly two stationary equilibria, as in Figure I: the lower one $n_L \in (\frac{1}{8},\frac{1}{2})$, and the higher one $n_H \in (\frac{1}{2},1)$. To verify that n_L is locally stable, note that because $n_L > \frac{1}{8}$, therefore $\gamma_{12}(\delta n_L,n_L)/\gamma_{11}(\delta n_L,n_L) = 2n_L/(1 - n_L) > \frac{2}{7} > \frac{1}{5} = r$.

V. Economic Stability and Indeterminacy

The local stability properties of the "Euler" equation (9) are sufficient to derive our main observability results, which address the question of whether there exist equilibrium trajectories that converge upon the stationary equilibrium in question. But a positive answer to this observability question does not guarantee that every time the economy begins in a neighborhood of a stationary equilibrium employment will in fact converge to that stationary equilibrium. That is, it does not guarantee that each stationary equilibrium is stable in the economic sense.

Consider, for example, the phase diagram Figure II, corresponding to Figure I, in the case where n_L is locally stable. The economy could start with employment equal to n_L yet follow the trajectory starting at point A that approaches n_H. Likewise every time initial employment was near n_L, it could follow this same path to n_H. Because it converges to n_H, any trajectory that follows this path is an equilibrium trajectory. Furthermore, if, as drawn, this convergent path bends down underneath n_L, then there will be two equilibrium trajectories starting at such an initial point that converge to n_H; one that starts with $\dot{n} < 0$, the other with $\dot{n} > 0$. By the same token there may be an equilibrium trajectory starting at such an initial point that converges upon the degenerate equilibrium at

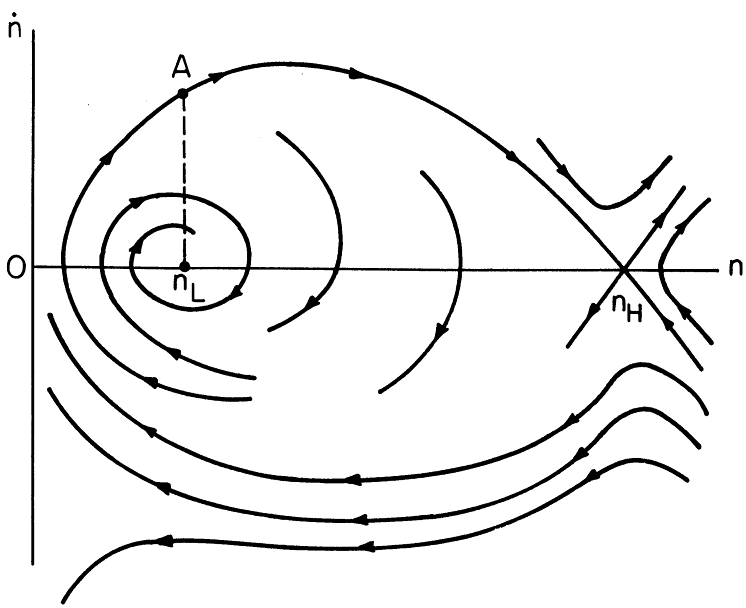

FIGURE II
Local Stability

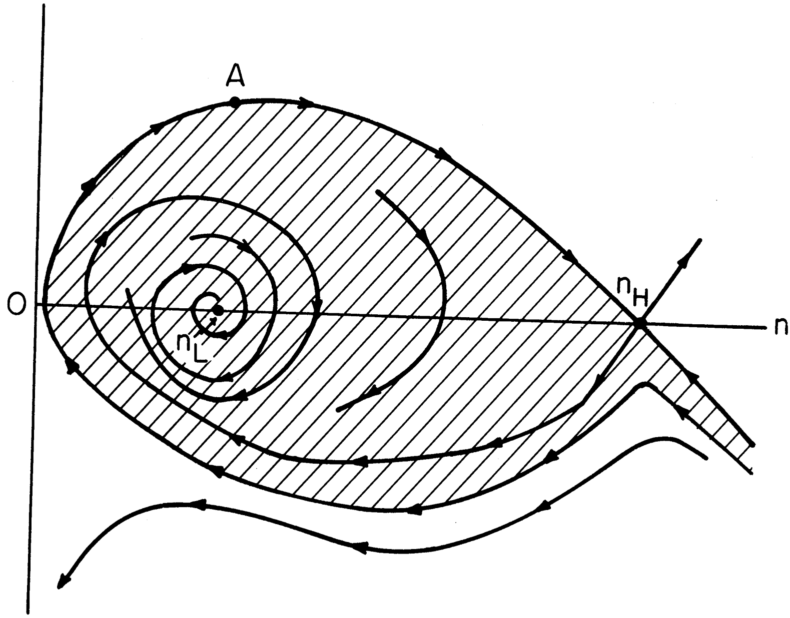

FIGURE III
Perfect Foresight Paths

the origin. Thus, even though n_L is locally stable under the "Euler" equation (9), it may not be economically stable.

Similar qualifications must be made with respect to the saddle-point equilibria. As in Figure II there may be an equilibrium path that goes through n_H and converges to the origin. Furthermore, if, as shown in Figure III, the stable path leading to n_H bends back under n_H as well as under n_L, there can be a continuum of equilibrium trajectories starting from any n in the neighborhood of n_H and converging upon n_L; specifically any trajectory starting in the shaded open loop of Figure III formed by the stable path.[3] (All paths diverging to the northeast violate "transversality.")

Thus, although a stationary equilibrium that is either a saddle-point or locally stable will be potentially observable, the economy may never approach it no matter how close it starts to it. It all depends upon which of the equilibrium trajectories is selected, and

3. Guckenheimer and Holmes [1983, pp. 290 ff] give examples of second-order systems with configurations like those of Figure II and III. Generally, there may be only one path to n_H.

our analysis says nothing about how the equilibrium trajectory is selected.[4]

There is one consideration that makes the locally stable low level equilibrium more likely to be observed than the high level saddlepoint. As Figures II and III illustrate, starting from any initial position that has some trajectories converging to n_L and some to n_H there will generally be a finite number converging on n_H but a continuum converging on n_L. Thus, a mechanism that selected an equilibrium trajectory at random would almost never select one converging on n_H.

We hasten to add, however, that the existence of a continuum of equilibrium trajectories raises the question of whether any of the trajectories will be followed. Along each of these trajectories every firm has perfectly predicted the employment decisions of every other firm. But in the absence of some mechanism for coordinating firms' predictions, there seems to be no way that an isolated firm could make such a prediction. It seems more likely that the firms would have to follow some trial and error procedure, which at most would converge asymptotically upon one of the equilibrium trajectories that we have described.[5] Our main point is just that, in the absence of any such trial and error analysis, we cannot rule out the possibility that the low level equilibrium of Figure I will be stable in an economically meaningful sense.

VI. Other Applications

The general form of our model (i.e. the differential equation (9) with the assumptions (2), (4), and (5)) is obviously applicable to a variety of problems other than the aggregate employment problem that has motivated our analysis. The important characteristic of

4. Efficiency aspects of equilibrium trajectories are complicated by the presence of both external economies and external diseconomies. Furthermore, in the search interpretation of the labor market, the wage-setting rule will generally induce inefficiencies, as shown by Mortensen [1982]. But we can be sure than when $\gamma_{12} > 0$ the equilibrium trajectories do not maximize any intertemporally additive social welfare function with a rate of discount r, for that would require the roots of (9) to sum to r. In the specific example analyzed in the previous version of this paper [1984], the trajectory that led monotonically to n_H Pareto-dominated every other trajectory from the same initial value of n.

5. Mussa [1984] points out other problematic aspects of models with a continuum of convergent solutions, most notably that they lack a dependency of current endogenous variables upon predictable future variations in exogenous variables. Whether this lack is a defect of such models, as Mussa asserts, or an empirical prediction that may be hard to reject in some contexts remains to be seen.

such a problem is that it can be described as the outcome of firms' intertemporal maximization, with current profits subject to an external economy sufficiently strong to make (4) hold, and eventually subject to diseconomies, either internal or external, that make (5) hold. What we have shown is that in such a context all stationary equilibria may be potentially observable, provided that there is an external diseconomy of expansion. Specifically, the equilibria at which $g'(n) < 0$ will exhibit the usual saddlepoint stability and each of the nonsaddle equilibria will be the limit of a continuum of perfect foresight trajectories going through the same initial activity level if $\gamma_{12}/\gamma_{11} > r$.

One context where the analysis might apply is that of the growth of a city. Agglomeration economies coexist with the natural limits imposed by the availability of land. Thus, not only might there be multiple equilibrium sizes to the city but each one might be observable.

Another example might be the market for any new product whose demand is ultimately limited by the availability of potential customers. The economy of scale could arise for a variety of reasons. Imitators can free ride on the increasing familiarity that raises demand when other firms operate on a larger scale, as IBM has been accused of doing with several products (see, for example, Burstein, [1984]). There is the likelihood that a service network or an auxiliary product market will develop for propane gas, for quadraphonic hi-fis that require special tapes, or for turbo-engined cars that require servicing by specially trained mechanics.[6] In all these cases the analysis suggests that one is likely to find observable low level equilibria.

The analysis may also be applicable to the development of the standard arrangement of keys in typewriters. As David [1985] has argued, the standard arrangement is demonstrably inferior to known alternatives, yet it persists. Here we have a case of an equilibrium with a low level of quality. David argues that external economies of scale have been important in explaining the stability of that low level. Manufacturers do not find it profitable to raise the quality until typists begin to learn on an improved arrangement, and typists do not find it worthwhile to learn until manufacturers change.

As a final example, consider the process of economic growth,

6. Markusen [1984] analyzes several examples of this sort.

which, according to countless writers from Adam Smith through Schumpeter is intimately connected with external economies of scale. Romer [1983] has analyzed this problem with a model similar to ours. His model, however, does not have the diseconomies of expansion required to render all stationary equilibria observable. Our analysis suggests that combining the limitations of finite natural resources together with the external economies considered by Romer can give rise to stable low level equilibria.

In all these examples, including the unemployment example of the previous sections, the multiplicity of equilibria obviously depends upon the firms' inability to internalize the external economy of scale. Similarly, the local stability of low level equilibria depends on the firms' inability to internalize the diseconomy of expansion. Such inability seems to make most sense in the macro examples of unemployment and economic growth, where internal diseconomies are likely to discourage the large-scale organization of the market in question under a few entrepreneurs. Casual empiricism suggests that internalization does occur in some of the micro examples cited, as where computer companies pay others to write software for their microcomputers. Still there is no a priori reason to believe that it is so extensive as to completely vitiate our analysis.

APPENDIX: DERIVATION OF (12)

Define

$$\phi(\ddot{n},\dot{n},n) = L_2(\dot{n},n,n) + rL_1(\dot{n},n,n) - L_{11}(\dot{n},n,n)\ddot{n} - L_{12}(\dot{n},n,n)\dot{n} - L_{13}(\dot{n},n,n)\dot{n}.$$

Equation (9) forces $\phi(\ddot{n},\dot{n},n) = 0$. Note that

$$\phi_1(0,0,n) = -L_{11}(0,n,n)$$

$$\begin{aligned}\phi_2(0,0,n) &= L_{12}(0,n,n) + rL_{11}(0,n,n) - L_{111}(0,n,n)\ddot{n} - (L_{12}(0,n,n) \\ &\quad + L_{13}(0,n,n)) - (L_{121}(0,n,n) + L_{131}(0,n,n)) \\ &= rL_{11}(0,n,n) + L_{13}(0,n,n),\end{aligned}$$

$$\begin{aligned}\phi_3(0,0,n) &= L_{22}(0,n,n) + L_{23}(0,n,n) + rL_{12}(0,n,n) + rL_{13}(0,n,n) \\ &\quad - \ddot{n}(L_{112} + L_{113}) - \dot{n}[L_{122} + L_{123} + L_{132} + L_{133}] \\ &= L_{22}(0,n,n) + L_{23}(0,n,n) + r(L_{12}(0,n,n) + L_{13}(0,n,n)).\end{aligned}$$

Now use a first-order Taylor expansion around an equilibrium n^*, so that

$$\phi(0,0,n^*) = 0,$$

$$\begin{aligned} 0 &= \phi(\ddot{n},\dot{n},n) \simeq \phi(0,0,n^*) + \phi_1(0,0,n^*)(\ddot{n} - 0) \\ &\quad + \phi_2(0,0,n^*)(\dot{n} - 0) + \phi_3(0,0,n^*)(n - n^*) \\ &= -L_{11}(0,n,n)\ddot{n} + [rL_{11}(0,n,n) - L_{13}(0,n,n)]\dot{n} \\ &\quad + [L_{22}(0,n,n) + L_{23}(0,n,n) + r(L_{12}(0,n,n) \\ &\quad + L_{13}(0,n,n))](n - n^*). \end{aligned}$$

Multiplying both sides by (-1), we obtain (12).

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