What is Pr{Z≥20}?

by

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Abstract: A close approximation to the standard normal is given for large values.

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It is not uncommon to find t statistics exceeding ten in some regressions. It turns out to be difficult to look up values for the probabilities in such cases. Here is an easy heuristic.

Theorem: For z>0,

$$\frac{1}{\sqrt{2p}}\;\frac{z}{z^2+1}e^{-z^2/2} \le \frac{1}{\sqrt{2p}}\int_z^{\infty} e^{-x^2/2}\,dx \;\le\; \frac{1}{z\,\sqrt{2p}}e^{-z^2/2}\;. \quad 1$$

Proof: Note that

$$\int_z^{\infty} e^{-x^2/2}\;dx = \int_z^{\infty} e^{-x^2/2}\;\; x\;\frac{1}{x}\;dx \quad 2$$

$$= -\,e^{-x^2/2}\frac{1}{x}\Big|_z^{\infty} - \int_z^{\infty} e^{-x^2/2}\;\; x^{-2}\;\; dx \quad 3$$

$$= \frac{1}{z}\;e^{-z^2/2} - \int_z^{\infty} e^{-x^2/2}\;\; x^{-2}\;\; dx \quad 4 \qquad (1)$$

$$\ge \frac{1}{z}e^{-z^2/2}\; - z^{-2}\int_z^{\infty} e^{-x^2/2}\;\; dx\;. \quad 5 \qquad (2)$$

The right inequality in the Theorem follows from (1), while the left follows from (2). ■

Remark 1: For values of z exceeding 10, either bound represents an error of at most 1% of the actual value.

Remark 2: Note that $e^x = 10^{x/ln10} \approx 10^{0.4343x}$. Therefore,

$$\Pr\{Z\ge z\} \approx \frac{0.3989}{t}10^{-0.21715t^2}\,, 6$$

and,

$$\Pr\{Z\ge 20\} \approx 0.\frac{3989}{20}\,\_\,10^{-86.86} = 2.753\_10^{-89}\,. 7$$

Writing this as one chance in n, n exceeds the estimates for the number of electrons in the universe by 10,000 times.