Reprinted from JOURNAL OF ECONOMIC THEORY

Vol. 32, No. 2, April 1984
Printed in Belgium

Joint Search for Several Goods

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Received October 29, 1981; revised February 18, 1983

Joint search occurs when a buyer incurs a single cost to observe prices of several different goods. If the prices are drawn from a known joint distribution function, the optimal sequential strategy with no recall uses a reservation sum for any subset of items. When the observed prices total more than the corresponding reservation sum, not all goods will be bought and search continues for items not purchased. Thus, regions in the price space are associated with various buy-search decisions. The reservation sums, however, have properties analogous to those of the reservation price with search for one good. *Journal of Economic Literature* Classification Number: 026.

1. INTRODUCTION

In a recent paper, Burdett and Malueg [2] pose the following problem. Suppose an individual planning to buy grocery products has a list of n goods ($n \geqslant 2$) to be purchased. There are many stores at which all the goods are sold. The individual can visit a store at a cost and get an n-vector of prices, one price for each of the n goods. If each price vector observed can be viewed as a random draw from a known non-degenerate distribution, what search strategy minimizes the expected cost of purchasing the n goods?

They consider two cases: free recall and no recall. With free recall, there is no cost in going back to a store already visited. In that case, nothing is

* We are indebted to an anonymous referee for helpful suggestions for simplifying our presentation and in particular for the proof of Theorem 3, which is much simpler than our original proof.

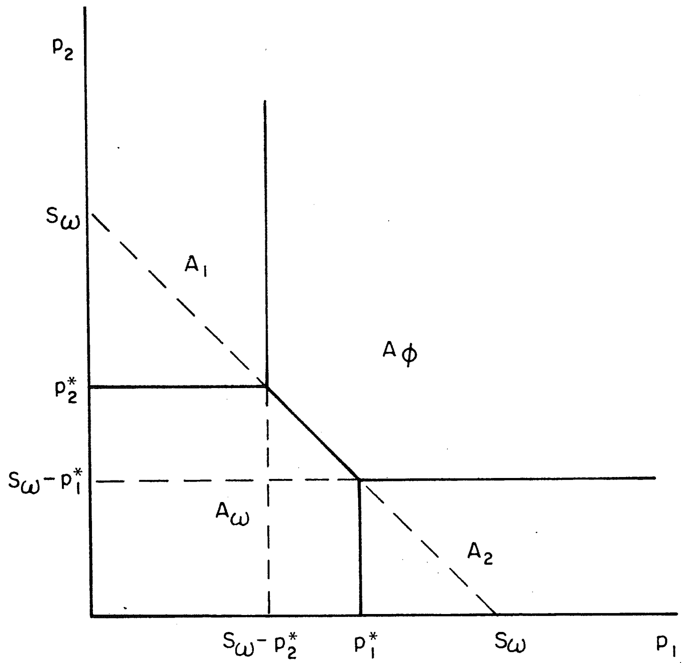

FIG. 1. Regions of price space corresponding to various buy-search decisions for two goods.

purchased until search is terminated. Then the individual returns costlessly to the stores which offer each of the goods at the lowest price observed. This is the case analyzed in much greater detail and yet, of the two cases considered, it seems to be less descriptive of how individuals actually shop for grocery products.

In the case of no recall, the individual has to decide whether to buy particular items while at each of the stores visited. Each decision corresponds to a region of the price space. If not all goods are bought, search is continued for the remaining items. Burdett and Malueg end their analysis of the no-recall case by characterizing the decision rule for 2 goods.

Figure 1 depicts the decision rule that Burdett and Malueg present for the two-good no-recall case in which a shopper has already drawn a price observation p_1 and p_2 at a cost c.[1] The decision whether to buy both goods, buy

[1] The same figure shows up in a paper by Adams and Yellen [1] on commodity bundling by a monopolist. They posit a distribution of reservation prices by buyers for each of two products. In mixed bundling the firm offers each good separately for p_i^* and both together for $p_B < p_1^* + p_2^*$. This partitions the buyers and is the strategy employed by the firm if it results in greater profits than pure bundling or simple monopoly pricing.

only one good, or buy neither depends on where the price observation lies relative to the reservation prices that a consumer would set if searching for each good alone at a cost c per observation (i.e., p_1^* and p_2^*) and relative to S_ω, which we shall call the *reservation sum.* The consumer makes the decision that results in the least additional expected cost. For example, in region A_ω where both goods are purchased, $p_1 + p_2$ is less than the additional expected cost of any of the other three actions. A draw in A_ϕ calls for the purchase of neither good and continue searching for both; in the region A_i the optimal strategy is to buy good i and continue searching for the other good. The term *joint search* means that the cost of search is incurred jointly for a set of goods.

We shall extend the analysis of the no-recall case to any number of goods. In a model of joint search with a large number of goods, the consumer's decision rule appears to be quite complex. With J goods, there are 2^J regions in the price space, each of which is associated with a different buy-search decision in terms of individual items.

The overall problem, however, is similar to the one-good search model. For every subset of items, there is reservation sum. Search will terminate if and only if the observed price vector falls within the reservation sum for every subset of items, such as the A_ω region pictured in Fig. 1 for two goods. With three goods there are seven boundaries to consider: an overall reservation sum, a reservation sum for any pair of goods, and a reservation price for each of the three goods. If any boundary is exceeded, search continues. And yet the outer plane (or hyperplane), i.e., the reservation sum for three goods (or for J goods), cuts any price axis at a value S_ω that also equals the total expected cost of searching for and buying all three (or all J) goods.

If, after visiting a store, the consumer chooses to purchase a subset of items, then the decision rule for the remaining subset has the same reservation-sum form with all the same boundaries on the new A_ω but within a reduced price space. The actual prices paid, once search has terminated, will all lie within the original A_ω region.

The theorems proved in the next section relate to the decision rules for any number J of goods. Theorem 1 demonstrates that each of the 2^J regions in price space, corresponding to each possible buy-search decision, is a convex set of prices. Theorem 2 shows the relationship between the total expected number of searches, the probabilities of drawing a price set in each region, and the expected number of searches for items not yet purchased.

Let c denote the cost of drawing a set of prices and let $S_\omega(c)$ denote the total expected cost of purchasing and searching

$$S_\omega(c) = Ep_\omega(c) + cEn_\omega(c) \tag{1}$$

where $Ep_\omega(c)$ is the expected cost of purchasing all J items and $En_\omega(c)$ is the expected number of draws. If there is a change in c, the effect on the total cost is

$$\frac{\partial S_\omega}{\partial c} = En_\omega > 0 \tag{2}$$

This is proved in Theorem 3.

Taking second derivatives

$$\frac{\partial^2 S_\omega}{\partial c^2} = \frac{\partial En_\omega}{\partial c} < 0 \tag{3}$$

This inequality is proved in Theorem 4.

As in the one-good case, an increase in c lowers the expected amount of search and raises the total expected cost. In addition, as c varies, there is a negative tradeoff between the expected purchase cost of the goods and the expected amount of search. The tradeoff becomes less steep at a higher level of expected search. In other words, as in one-good search models, there are positive and diminishing gains to the expected amount of search. Empirical tests of these predictions are reported in Carlson and Gieseke [3].[2]

Thus, in several important respects, a bundle of goods subject to joint search may be treated as if it were one good.

2. The General Case with No Recall

Suppose there are J goods, where J may be any positive integer. Define the set $\omega = \{1,..., J\}$. For any subset $\alpha \subseteq \omega$, the complement of α will be denoted $\bar{\alpha}$. Let S_α = minimum expected total cost of purchasing goods in α. Note that $S_\phi = 0$.

Given a price observation $p = (p_1,..., p_J)$, the expected cost associated with buying the subset α and continuing search in an optimal fashion for the subset $\bar{\alpha}$ is $\sum_{i \in \alpha} p_i + S_{\bar{\alpha}}$. Let

$$A_\alpha = \left\{ p \in R_+^J \mid (\forall \gamma \subseteq \omega) \left(\sum_{i \in \alpha} p_i + S_{\bar{\alpha}} \leqslant \sum_{i \in \gamma} p_i + S_{\bar{\gamma}} \right) \right\} \tag{4}$$

A_α is the region of the price space in which the least-cost strategy is to buy the set α and plan to look elsewhere for the remaining items. This is

[2] These relationships can be seen by first differentiating both sides of (1) with respect to c: $\partial S_\omega/\partial c = \partial Ep_\omega/\partial c + c(\partial En_\omega/\partial c) + En_\omega$. Note from (2) that $\partial S_\omega/\partial c = En_\omega$ so that $(*)\ \partial Ep_\omega/\partial En_\omega = (\partial Ep_\omega/\partial c)/(\partial En_\omega/\partial c) = -c < 0$. Taking the second derivative: $(**)\ \partial^2 Ep_\omega/\partial En_\omega^2 = (\partial(\partial Ep_\omega/\partial En_\omega)/\partial c)/(\partial En_\omega/\partial c) = -1/(\partial En_\omega/\partial c) > 0$ since by (3), $\partial En_\omega/\partial c < 0$. Carlson and Gieseke using proxies for Ep_ω and En_ω from panel data on grocery purchases find strong support for $(*)$ and marginally significant support for $(**)$.

illustrated in Fig. 1 for the two-good case, with $\omega = \{1, 2\}$. The following theorem indicates a feature of these subspaces that follows from their definition in (4).

THEOREM 1. *A_α is a convex subset of R^J_+.*

Proof. Suppose $p^0, p^1 \in A_\alpha$. Then for all $\gamma \subseteq \omega$,

$$\sum_{i\in\alpha} p_i^0 + S_{\bar\alpha} \leqslant \sum_{i\in\gamma} p_i^0 + S_{\bar\gamma}; \qquad \sum_{i\in\alpha} p_i^1 + S_{\bar\alpha} \leqslant \sum_{i\in\gamma} p_i^1 + S_{\bar\gamma}$$

Thus, if $\lambda \in [0, 1]$

$$\begin{aligned}\sum_{i\in\alpha} (\lambda p_i^0 + (1-\lambda) p_i^1) + S_{\bar\alpha} &= \lambda \left(\sum_{i\in\alpha} p_i^0 + S_{\bar\alpha}\right) + (1-\lambda)\left(\sum_{i\in\alpha} p_i^1 + S_{\bar\alpha}\right)\\ &\leqslant \lambda \left(\sum_{i\in\gamma} p_i^0 + S_{\bar\gamma}\right) + (1-\lambda)\left(\sum_{i\in\gamma} p_i^1 + S_{\bar\gamma}\right)\\ &= \sum_{i\in\gamma} (\lambda p_i^0 + (1-\lambda) p_i^1) + S_{\bar\gamma}\end{aligned}$$

Therefore $\lambda p^0 + (1-\lambda) p^1 \in A_\alpha$, so A_α is convex. Q.E.D.

Define the measure dF over R^J_+ by $\int h(p) f(p)\, dp = \int h(p)\, dF$ for any h: $R^J_+ \to R$, where $f(p)$ is the density function of $p \in R^J$ and f is assumed to be continuous. Define the probability of a price observation in A_α by

$$\beta_\alpha = \int_{A_\alpha} dF$$

Let En_α = expected number of draws to purchase set α using the optimal policy and Ep_α = expected payment for purchasing set α. For the null set, $Ep_\alpha = En_\alpha = 0$.

THEOREM 2. *$En_\omega = (1-\beta_\phi)^{-1}(1 + \sum_{\phi\neq\alpha} \beta_\alpha En_{\bar\alpha})$.*

Proof. Evidently, $En_\omega = \sum_{\alpha\subseteq\omega} \beta_\alpha(1 + En_{\bar\alpha})$. But $\sum_{\alpha\subseteq\omega}\beta_\alpha = 1$, so

$$En_\omega = \sum_{\phi\neq\alpha} \beta_\alpha En_{\bar\alpha} + \beta_\phi En_\omega + 1$$

Thus,

$$En_\omega = (1-\beta_\phi)^{-1}\left(\sum_{\phi\neq\alpha} \beta_\alpha En_{\bar\alpha} + 1\right). \qquad \text{Q.E.D.}$$

For example, with two goods:

$$En_\omega = (1 + \beta_1 En_2 + \beta_2 En_1)/(1-\beta_\phi)$$

This can be explained intuitively. β_ϕ is the probability of not buying anything on a particular draw. Thus, the expected number of searches until at least one good is purchased is $1/(1-\beta_\phi)$. The proportion of the time that only good 1 is purchased first is $\beta_1/(1-\beta_\phi)$ and the expected number of additional searches for good 2 is En_2. A similar explanation holds for the term $\beta_2 En_1/(1-\beta_\phi)$.

THEOREM 3. *If f is continuous, then $\partial S_\omega/\partial c$ exists and $\partial S_\omega/\partial c = En_\omega$.*

Proof. For any c and c'

$$S_\omega(c') \leqslant Ep_\omega(c) + c'En_\omega(c) \tag{5}$$

since the right-hand side of (5) is the expected cost of using the optimal search strategy for search costs c but when search costs are actually c'. Then add and subtract $cEn_\omega(c)$ on the right-hand side of (5) and use (1) to get

$$S_\omega(c') \leqslant S_\omega(c) + (c'-c)\, En_\omega(c) \tag{6}$$

By a similar argument

$$S_\omega(c) \leqslant S_\omega(c') - (c'-c)\, En_\omega(c') \tag{7}$$

Assume, without loss of generality, that $c' > c$. Then from (6) and (7)

$$En_\omega(c') \leqslant \frac{S_\omega(c') - S_\omega(c)}{c'-c} \leqslant En_\omega(c)$$

Since A_α changes continuously with c (and hence so does β_α), $En_\omega(c') \to En_\omega(c)$ as $c' \to c$. Therefore $\partial S_\omega/\partial c = En_\omega$. Q.E.D.

Note that (6) also establishes that $S_\omega(c)$ is concave in c, since it is supported by a linear function with slope $En_\omega(c)$. It remains to show that $En_\omega(c)$ is differentiable with a strictly negative derivative.

LEMMA 1. *If f is continuous, then dEn_ω/dc exists.*

Proof. Recall that $S_\alpha(c)$ is differentiable and A_α is defined by $\sum_{i\in\alpha} p_i + S_{\bar{\alpha}} = \sum_{i\in\gamma} p_i + S_{\bar{\gamma}}$ hyperplanes. It follows that

$$\frac{\partial}{\partial S_\gamma} \int_{A_\alpha} f(x)\, dx$$

exists for any continous f. See Sagan [4, p. 542]. By Stokes' theorem, then

$$\frac{d\beta_\alpha}{dc} = \sum_{\gamma\subseteq\omega} \frac{dS_\gamma}{dc} \frac{\partial}{\partial S_\gamma} \int_{A_\alpha} dF$$

exists. Consequently, En_α is differentiable for $\alpha \neq \omega$. In view of Theorem 2, then, En_ω is differentiable. Q.E.D.

The value of $\partial\beta_\alpha/\partial c$ is given by the value of f on the borders of A_α times the differential directed magnitude of change in the borders. In the following, we consider the derivatives on each border of A_α. Denote this

$$\frac{\partial\beta_\alpha}{\partial c} = \sum_{\gamma\neq\alpha} \left.\frac{\partial\beta_\gamma}{\partial c}\right|_{B_{\alpha\gamma}} \tag{8}$$

where $B_{\alpha\gamma} = A_\alpha \cap A_\gamma$ is the border between A_α and A_γ.

LEMMA 2.

$$En_{\bar{\alpha}} \left.\frac{\partial\beta_\alpha}{\partial c}\right|_{B_{\alpha\gamma}} + En_{\bar{\gamma}} \left.\frac{\partial\beta_\gamma}{\partial c}\right|_{B_{\alpha\gamma}} \leqslant 0.$$

Proof. By definition, for $p \in B_{\alpha\gamma}$

$$\sum_{i\in\alpha} p_i + S_{\bar{\alpha}} = \sum_{i\in\gamma} p_i + S_{\bar{\gamma}}$$

Then p will be in the interior of A_γ after an infinitesimal increase in c if and only if

$$\partial S_{\bar{\alpha}}/\partial c > \partial S_{\bar{\gamma}}/\partial c$$

or, equivalently by Theorem 3,

$$En_{\bar{\alpha}} > En_{\bar{\gamma}} \tag{9}$$

Assuming that (9) is true:

$$\left.\frac{\partial\beta_\gamma}{\partial c}\right|_{B_{\alpha\gamma}} \geqslant 0, \qquad \left.\frac{\partial\beta_\alpha}{\partial c}\right|_{B_{\alpha\gamma}} \leqslant 0,$$

and, without loss of generality,

$$\begin{aligned} En_{\bar{\alpha}} \left.\frac{\partial\beta_\alpha}{\partial c}\right|_{B_{\alpha\gamma}} + En_{\bar{\gamma}} \left.\frac{\partial\beta_\gamma}{\partial c}\right|_{B_{\alpha\gamma}} &\leqslant En_{\bar{\gamma}} \left.\frac{\partial\beta_\alpha}{\partial c}\right|_{B_{\alpha\gamma}} + En_{\bar{\gamma}} \left.\frac{\partial\beta_\gamma}{\partial c}\right|_{B_{\alpha\gamma}} \\ &= En_{\bar{\gamma}} \left.\frac{\partial(\beta_\alpha + \beta_\gamma)}{\partial c}\right|_{B_{\alpha\gamma}} = 0. \end{aligned}$$

Q.E.D.

An interesting implication of the proof is that the effect of incrementing c is unequivocally to induce the buyer to shift away from high search areas at each border.

LEMMA 3.

$$\sum_{\alpha\neq\phi}\frac{\partial\beta_\alpha}{\partial c}En_{\bar\alpha}\leqslant En_\omega\frac{\partial(1-\beta_\phi)}{\partial c}.$$

Proof.

$$\begin{aligned}\sum_{\alpha\neq\phi}\frac{\partial\beta_\alpha}{\partial c}En_{\bar\alpha}+En_\omega\frac{\partial\beta_\phi}{\partial c}&=\sum_{\alpha}\frac{\partial\beta_\alpha}{\partial c}En_{\bar\alpha}\\&=\sum_{\alpha}\sum_{\gamma\neq\alpha}\frac{\partial\beta_\alpha}{\partial c}\bigg|_{B_{\alpha\gamma}}En_{\bar\alpha}\\&=\sum_{\gamma}\sum_{\alpha\neq\gamma}\frac{\partial\beta_\alpha}{\partial c}\bigg|_{B_{\alpha\gamma}}En_{\bar\alpha}\\&=\frac{1}{2}\sum_{\gamma}\sum_{\alpha\neq\gamma}\left[\frac{\partial\beta_\alpha}{\partial c}\bigg|_{B_{\alpha\gamma}}En_{\bar\alpha}+\frac{\partial\beta_\gamma}{\partial c}\bigg|_{B_{\alpha\gamma}}En_{\bar\gamma}\right]\quad\text{the }\tfrac{1}{2}\text{ because each border is counted twice}\\&\leqslant 0\quad\text{by Lemma 2.}\end{aligned}$$

Thus, as

$$-\frac{\partial\beta_\phi}{\partial c}=\frac{\partial}{\partial c}(1-\beta_\phi),$$

$$\sum_{\alpha\neq\phi}\frac{\partial\beta_\alpha}{\partial c}En_{\bar\alpha}\leqslant En_\omega\frac{\partial(1-\beta_\phi)}{\partial c}\qquad\text{Q.E.D.}$$

Let $\operatorname{supp} f=\{p\mid f(p)>0\}$.

THEOREM 4. *If there exist α, γ such that* $\operatorname{supp} f\cap B_{\alpha\gamma}\neq\phi$, *then*

$$\frac{\partial^2S_\omega}{\partial c^2}=\frac{\partial En_\omega}{\partial c}<0$$

Proof. By induction. Suppose $\alpha\subset\omega$ $(\alpha\neq\omega)$ implies $\partial En_\alpha/\partial c<0$. Known for $\alpha=\{1\}$.

$$En_\omega=(1-\beta_\phi)^{-1}\left[1+\sum_{\phi\neq\alpha}\beta_\alpha En_{\bar\alpha}\right]$$

$$\begin{aligned}\frac{\partial En_\omega}{\partial c}&=(1-\beta_\phi)^{-1}\left[-En_\omega\frac{\partial(1-\beta_\phi)}{\partial c}+\sum_{\phi\neq\alpha}\beta_\alpha\frac{En_{\bar\alpha}}{\partial c}+\sum_{\phi\neq\alpha}\frac{\partial\beta_\alpha}{\partial c}En_{\bar\alpha}\right]\\&\leqslant(1-\beta_\phi)^{-1}\sum_{\phi\neq\alpha}\beta_\alpha\frac{\partial En_{\bar\alpha}}{\partial c}\leqslant 0\qquad\begin{array}{l}\text{(by Lemma 3)}\\\text{(by induction hypothesis).}\end{array}\end{aligned}$$

Inspection of the lemmas reveals that equality holds if and only if the evaluation of f on $B_{\alpha\gamma}$ is zero almost everywhere. In particular, if f is continuous, then equality holds only if f is zero on all boundaries. Q.E.D.

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium