January 26, 2007

The Gains from Trade Under Fixed Price Mechanisms

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Abstract: The Myerson—Satterthwaite Theorem (1983) demonstrates that information must interfere with efficient exchange when a buyer and seller have independently distributed private information and there is a nontrivial exchange problem. I bound the losses associated with private information based on a simple characteristic for of the distributions. When the median of the seller's cost is less than the median of the buyer's value, the maximal loss is one half of the full information gains from trade. As a fraction of the possible gains from trade, the losses tend to be greatest precisely when seller is likely to have high costs, and thus when the overall gains from trade are small.

In a hugely important article, Roger Myerson and Mark Satterthwaite demonstrate that if a buyer and seller have independently distributed, privately observed valuations for an indivisible good, and there is no source of external subsidy, then it is not possible to arrange efficient trade. This is an important theorem because it shows that information is a barrier to efficient exchange, and indeed the distribution of information acts like a constraint analogous to a resource or scarcity constraint. Moreover, the problem posed is a conventional bilateral bargaining problem, with the reasonable assumption that the seller is uncertain about the buyer's value and vice-versa. The only distributional assumption made is that there is a non-trivial allocation problem, so that sometimes trade is efficient and sometimes it is not.

Private information gives rise to informational rents, because a privately informed individual can pretend to have different information than they possess. In other words, private information can be used strategically to benefit its possessor. Moreover, the size of the informational rents depends on the amount of exchange the information can influence. The nature of the Myerson—Satterthwaite Theorem is a proof that under efficiency the informational rents collectively exceed the total gains from trade, so that any efficient mechanism requires subsidies from outside.

How serious are informational constraints in practice? A sizeable literature, discussed below, seeks to bound the effects of informational constraints. This paper contributes to that literature by showing that, if the median of the buyer's value exceeds the median of the seller's cost, the efficiency loss is at most half of the possible gains from trade. The paper also provides a more general formula and suggests that losses are large precisely in the circumstances that the total gains are small.

Many authors have provided limits to the efficiency loss associated with the Myerson—Satterthwaite Theorem may be mitigated by other considerations. First, if there is a thick market, the per trader efficiency loss is small, going to zero at a rate of the inverse of the lesser of the number of buyers and sellers. Thus, large markets perform close to the supply and demand model of price-taking behavior, while thin markets may have substantial informational inefficiency. The most prominent paper in this literature is Satterthwaite and Williams (1989), but McAfee (1992) provides the easiest proof of the rate of convergence.

Second, Ausubel and Deneckere (1993) show that if trades can unfold over time, efficient exchange can eventually be reached, and moreover reaching efficiency is part of the optimal solution to the bilateral bargaining problem. Efficient trade is eventually obtained because delay is costly to the agents; note there is no gain in efficiency relative to the original Myerson—Satterthwaite Theorem, but rather a characterization of the form that the inefficiency takes. Nevertheless, it is an important addition because Ausubel and Deneckere show that the inefficiency doesn't persist indefinitely.

Third, the Myerson—Satterthwaite Theorem concerns a discrete good. In many circumstances (e.g. a specific car, rights to publish a novel), the good is discrete. But in other situations (e.g. labor supplied by a union, proportion of installations of a given operating system by a computer manufacturer), the good is more reasonably modeled as continuous. In this situation, McAfee (1991) shows that efficient exchange may be possible and provides an exact characterization of when it is. Note that the Myerson—Satterthwaite Theorem can be taken as formulated as a limiting environment of the continuous model, so that in some continuous circumstances, efficient exchange remains impossible.

Fourth, Cremer and McLean (1988) (in the finite value environment) and McAfee and Reny (1992) (in the continuous value environment) show that correlation may permit efficient exchange. Efficiency arises in both papers because information held by one party can be used to extract rents from the other party by way of a "participation charge," thereby eliminating nearly all of the informational rents the parties expect. This demonstration disables the Myerson-Satterthwaite Theorem directly, permitting efficient exchange. McAfee and Reny, however, go further and provide a closed form mechanism that conditions the payment made by the buyer with the seller's information, extending the circumstances in which efficient trade is feasible even beyond that available with participation charges which don't depend on whether a sale ultimately occurs.

In contrast to these four mitigations, this paper considers the standard Myerson—Satterthwaite exchange of a discrete good under independently distributed values, and investigates the size of the loss arising from private information.

The Theory

There is a seller with an indivisible good, and one potential buyer. The seller values the good at c. The seller knows c but from the buyer's perspective, c is a realization of a continuous distribution F which has density f. The buyer values the good at v; this value is known to the buyer but viewed by the seller as a draw from a continuous distribution G with density g. The supports of both F and G are assumed to be contained in $[0,\infty)$. The decision to trade is non-trivial, meaning that an interval is contained in the intersection of the supports. Finally it is assumed that both distributions have well-defined means.

Under full information, trade would take place whenever $v>c$, which gives gains from trade of

$$M = \int_0^\infty \int_0^v (v-c)g(c)dc f(v)dv = \int_0^\infty \left((v-c)G(c) \Big|_0^v + \int_0^v G(c)dc \right) f(v)dv$$

$$= \int_0^\infty \left(\int_0^v G(c)dc \right) f(v)dv = -(1-F(v)) \int_0^v G(c)dc \Bigg|_0^\infty + \int_0^\infty (1-F(v))G(v)dv$$

$$= \int_0^\infty (1-F(v))G(v)dv .$$

Note that existence of the mean of F is sufficient to insure the integrals converge, because $\int_0^v G(c)dc \le v$.

Suppose a given price p is imposed on the buyer and seller independently of the realization of the values, and trade takes place only if they both agree to trade at the price p. Then trade occurs whenever $v>p>c$. Under this mechanism there will be some circumstances where efficient trade fails to arise. The gains from trade are

$$GFT(p) = \int_p^\infty \int_0^p (v-c)g(c)dcf(v)dv = \int_p^\infty \int_0^p (v-p+p-c)g(c)dcf(v)dv$$

$$= \int_p^\infty \int_0^p (v-p)g(c)dcf(v)dv + \int_p^\infty \int_0^p (p-c)g(c)dcf(v)dv$$

$$= \int_p^\infty (v-p)G(p)f(v)dv + (1-F(p))\int_0^p (p-c)g(c)dc$$

$$= G(p)\int_p^\infty (v-p)f(v)dv + (1-F(p))\left((p-c)G(c) \Big|_0^p + \int_0^p G(c)dc \right)$$

$$= G(p)\left(-(v-p)(1-F(v)) \Big|_p^\infty + \int_p^\infty (1-F(v))dv \right) + (1-F(p))\int_0^p G(c)dc$$

$$= G(p)\int_p^\infty (1-F(v))dv + (1-F(p))\int_0^p G(c)dc .$$

It is now straightforward to prove the main theorem.

Theorem: Fix a price p. Then $\dfrac{GFT(p)}{M} \ge \min\{G(p), 1-F(p)\}$.

Proof: $GFT(p) = G(p)\int_p^\infty (1-F(v))dv + (1-F(p))\int_0^p G(c)dc$

$$\geq Min\{G(p),(1-F(p))\}\left(\int_p^\infty (1-F(v))dv + \int_0^p G(c)dc\right)$$

$$\geq Min\{G(p),(1-F(p))\}\left(\int_0^p (1-F(c))G(c)dc + \int_p^\infty G(v)(1-F(v))dv\right) \qquad (0 \leq F, G \leq 1)$$

$$= Min\{G(p),(1-F(p))\}\left(\int_0^\infty G(v)(1-F(v))dv\right) = Min\{G(p),(1-F(p))\}M \text{ . Q.E.D.}$$

This theorem has a striking corollary:

Corollary 1*: Suppose the median of the buyer's distribution exceeds the median of the seller's distribution. Then the gains from trade using any fixed price between the two medians is at least half of the maximum possible gains from trade.*

The corollary holds because for any price between the two medians, $Min\{G(p),(1-F(p))\} \geq ½$. This is a nice result because the property that the median of the buyer's value exceeds the median of the seller's cost is an economically reasonable assumption in many settings. In particular, whenever the buyer's value first-order stochastically dominates the seller's cost, the condition will be met.

The lower bound in the theorem is tightest when $G(p^*)+F(p^*)=1$, leading to the following corollary.

Corollary 2*: Let p^* solve $G(p^*)+F(p^*)=1$. Then* GFT(p*) $\geq G(p^*)M$.

How well do these bounds do? There are really four comparisons: the Corollary 2 bound, the best fixed price mechanism, the maximum possible given the incentive constraints, and the maximum possible under full information. To illustrate the bounds, I consider exponential distributions on $[0,\infty)$, and consider the proportion of the maximum possible gains achieved under the bound and the two mechanisms. These ratios depend only on the ratio of the means of the distributions, so without loss of generality given the exponential assumption, I set the mean of buyer's distribution F to be one, and the mean of the seller's distribution to be $1/a$. Appendix 1 and 2 sets out the calculations of the optimal mechanism given the incentive constraints and the specialization to the exponential distribution.

When the mean of the seller's distribution exceeds the mean of the buyer's distribution (and hence the same is true about the median), the Corollary 2 bound does a poor job. However, seller on average has lower cost than the buyer, both the mechanisms are relatively efficient, and the bound does reasonably well, as illustrated in Figure 1.

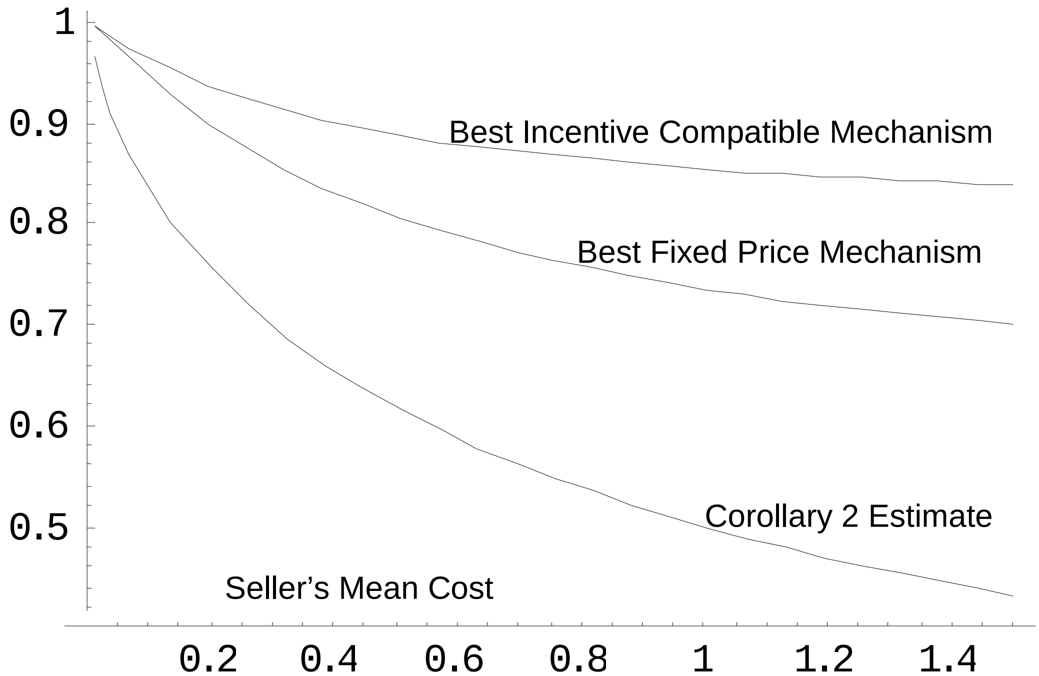

Figure 1: Proportion of the Possible Gains from Trade Using Incentive Compatible Mechanisms

An important conclusion is that the efficiency loss, as a share of the total surplus available, associated with asymmetries of information tend to be small when the gains from trade are large, and tend to be large when the gains from trade are small. In particular, an increase in the seller's cost distribution will reduce both the efficiency as a percentage and the total gains from trade. Thus, in the circumstances where from a social perspective it matters most, asymmetries of information tend to cause the least harm. Conversely, asymmetries of information tend to cause a lot of harm precisely when the stakes are small.

For specific distributions like the exponential distribution, the bounds in the corollary are lower than necessary, but the fact that they depend so weakly on the distribution, and require no hazard conditions like McAfee (2002) and Hoppe, Moldovanu and Sela (2006) require, is a major advantage. In principle hazard conditions could tighten the bound.

Conclusion

Fixed price mechanisms have major advantages. They are simple and clearly incentive compatible. More importantly, players have simple, familiar and obvious optimal strategies, so that the scope for mistakes and errors are modest.

This papers shows that fixed price mechanisms do reasonably well in general. First, under the condition that the buyer's distribution dominates the seller's distribution, a fixed price mechanism gets at least half of the full information optimum. It is important to realize that this target is not achievable; every incentive compatible mechanism without outside subsidies does worse than the full information optimum. For the exponential distribution, a fixed price mechanism obtains at least 86% of the gains from trade achieved by any incentive compatible mechanism, which is 73% of the full information level.

While the losses are significant, they tend to arise when the seller's costs tend to be large relative to the buyer's value, and thus the probability that trade is efficient is small. Therefore, the exact circumstances when private information substantially constrains trade are those when trade isn't so important.

Appendix 1: The Effect of Incentive Constraints

This appendix provides a relatively simple derivation of the maximal gains from trade given the informational constraint, using the revelation principle as derived by Myerson (1983). The form of the proof is to characterize the buyer and seller surplus and the gains from trade all as a function of the probability of trade, and then maximize the gains from trade subject to the constraint that the sum of buyer and seller surplus does not exceed the gains from trade.

The mechanism that maximizes the gains from trade assigns a probability of trade $p(v,c)$ when the buyer's value is v and the seller's cost is c. In addition, let $y_b(v)$ be the buyer's expected payment given a report of v. The buyer's expected utility, which maximizes over the buyer's report *r,* is

$$u(v) = \max_r \int_0^\infty p(r,s)(v - y_b(r))g(s)ds .$$

From the envelope theorem and incentive compatibility,

$$u'(v) = \int_0^\infty p(v,s)g(s)ds \geq 0 .$$

Because u is non-decreasing, the buyer's willingness to participate is equivalent to $u(0) \geq 0$. Integrating by parts, the buyer's expected surplus is

$$Eu = \int_0^\infty u(v)f(v)dv = u(0) + \int_0^\infty u'(v)(1-F(v))dv = u(0) + \int_0^\infty \int_0^\infty p(v,s)g(s)ds(1-F(v))dv$$

$$= u(0) + \int_0^\infty \int_0^\infty p(v,s)\frac{1-F(v)}{f(v)}g(s)ds\ f(v)dv .$$

A very similar development, adjusting for the immediate observation that the seller's utility is decreasing in their type, shows the seller's utility is

$$E\pi = \pi(\infty) + \int_0^\infty \int_0^\infty p(v,s)\frac{G(s)}{g(s)}g(s)ds\ f(v)dv .$$

Finally, the gains from trade are

$$GFT = \int_0^\infty \int_0^\infty p(v,s)(v-s)g(s)ds\ f(v)dv .$$

Maximizing the gains from trade subject to the incentive constraints and no outside subsidies reduces to choosing p to maximize GFT subject to $GFT \geq \mathrm{E}u + \mathrm{E}\pi$.[1] Note immediately that setting $u(0)=\pi(\infty)=0$ weakens the no subsidies constraint.

The Myerson—Satterthwaite Theorem demonstrates that the unconstrained maximization of GFT fails the constraint. Let λ be the Lagrangian multiplier on the constraint. The Lagrangian expression is:

$$L = \int_0^\infty \int_0^\infty p(v,s)\left(v-s+\lambda\left(v-s-\frac{G(s)}{g(s)}-\frac{1-F(v)}{f(v)}\right)\right) g(s)dsf(v)dv .$$

If we let $\beta = \dfrac{\lambda}{1+\lambda}$, the solution comes in the form

$$p(v,s) = \begin{cases} 1 & v-s > \beta\left(\dfrac{G(s)}{g(s)}+\dfrac{1-F(v)}{f(v)}\right) \\ 0 & v-s < \beta\left(\dfrac{G(s)}{g(s)}+\dfrac{1-F(v)}{f(v)}\right) \end{cases} .$$

Moreover, the value of β is such that $GFT = \mathrm{E}u + \mathrm{E}\pi$. Finally, since the gains from trade are falling in β, the desired value is the first value of β such that $GFT = \mathrm{E}u + \mathrm{E}\pi$.[2]

Appendix 2: The Exponential Derivation

Let $F(v) = 1 - e^{-v}$ and $G(v) = 1 - e^{-av}$. Then

$$M = \int_0^\infty (1-F(v))G(v)dv = \int_0^\infty e^{-v}(1-e^{-av})dv = \frac{a}{1+a} .$$

The gains from trade at price p are

[1] There are two additional constraints that insure the first order conditions characterize utility maximizations: $\int_0^\infty p(v,s)g(s)ds$ is increasing in v and $\int_0^\infty p(v,s)f(v)dv$ is deceasing in s. The constraints will be handled by imposing regularity conditions sufficient to insure that the unconstrained solution meets these constraints. The ironing procedure for handling these constraints is straightforward due to the linearity in p.

[2] It is readily verified that if the usual hazard rate conditions are imposed, p is increasing in v and decreasing in s and thus satisfies the remaining conditions.

$$GFT(p) = G(p)\int_{p}^{\infty}(1-F(v))dv + (1-F(p))\int_{0}^{p}G(c)dc$$

$$= \frac{e^{-(1+a)p}}{a}\left(1-a+e^{ap}(a(1+p)-1)\right).$$

The approximation from Corollary 2 involves the solution to $G(p^*)+F(p^*)=1$, or

$e^{-p^*}+e^{-ap^*}=1$, and this implies $\frac{GFT(p^*)}{M} \geq e^{-p^*}$.

Then

$$p(v,s) = \begin{cases} 1 & v > s+\lambda\left(\frac{1}{a}\left(e^{as}-1\right)+1\right) \\ 0 & v < s+\lambda\left(\frac{1}{a}\left(e^{as}-1\right)+1\right) \end{cases}.$$

Let $q(s) = s+\lambda\left(\frac{1}{a}\left(e^{as}-1\right)+1\right)$, so that a trade takes place whenever $v>q(s)$.

Note that the gains from trade are decreasing in λ because the probability of trade is falling as λ increases. Thus, the solution for the maximal incentive compatible can be computed by finding the minimum value of λ for which $GFT = \mathrm{E}u + \mathrm{E}\pi$. The actual values are

$$GFT = \int_0^\infty\int_0^\infty p(v,s)(v-s)f(v)dv\, g(s)ds = \int_0^\infty \int_{q(s)}^\infty (v-s)e^{-v}dv\, ae^{-as}ds.$$

$$Eu + E\pi = \int_0^\infty\int_0^\infty p(v,s)\left(\frac{1-F(v)}{f(v)}+\frac{G(s)}{g(s)}\right)f(v)dvg(s)ds$$

$$= \int_0^\infty \int_{q(s)}^\infty \left(1+\frac{e^{as}-1}{a}\right)e^{-v}dv\, ae^{-as}ds = \int_0^\infty ae^{-as}\left(1+\frac{e^{as}-1}{a}\right)e^{-q(s)}dv\, ds.$$

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