Reprinted from JOURNAL OF ECONOMIC THEORY

Vol. 53, No. 1, February 199
Printed in Belgium

Efficient Allocation with Continuous Quantities

R. PRESTON MCAFEE*

University of Western Ontario, London, Ontario, Canada N6A 5C2 and California Institute of Technology, Pasadena, California 91125

Received February 24, 1989; revised February 22, 1990

The inefficiency of allocation mechanisms in the presence of bilateral asymmetric information is reconsidered in an environment with continuous quantities. The result of Myerson and Satterthwaite is proved in this environment under the condition that zero trade is efficient if the highest cost seller (or the lowest value buyer) appears. In addition, if this condition fails, there may exist mechanisms implementing efficient allocations. The problem of "hidden endowments" is considered, where any agent may be either a buyer or seller, depending on the realization of the privately observed information. In this environment, it is often possible to arrange efficient trades. Ex ante asymmetries, rather than interim asymmetries, tend to prevent efficient allocations. *Journal of Economic Literature* Classification Number: 026.

Myerson and Satterthwaite [14] consider the design of a mechanism to arrange an efficient allocation, when the valuations of a commodity are private information. In particular, they consider the environment where an indivisible unit of a commodity is possessed by a seller who values the item at s, and a single buyer values the item at t, and s and t are known only to the seller and buyer, respectively. The buyer (seller) views $s(t)$ as being chosen from some density $f(g)$, and s and t are independently distributed. Their result is that, if the intersection of the supports of f and g contains an interval (that is, there is a nontrivial decision of whether to trade or not), then there is no mechanism which arranges efficient trades and breaks even on average.

The Myerson–Satterthwaite result would lead one to believe that efficient trading in a bilateral asymmetric information environment is not possible, provided the innocuous "intersecting supports" condition is satisfied. However, I shall show that the condition leading to the impossibility of efficient trades is much less innocuous in two variants of the Myerson–Satterthwaite framework. The variants analyzed are more like the standard

* I thank Philip Reny, Alan Slivinski, an associate editor and an anonymous referee of this journal for their remarks.

0022-0531/91 $3.00

textbook demand model (e.g., Alchian and Allen, [1, pp. 24–31]). In particular, I consider first a model with continuous quantities, so that efficiency requires not only a decision of when to trade, as in Myerson–Satterthwaite, but also a decision of how much to trade.

The condition characterizing implementability of efficient allocations is reasonably straightforward. Consider the total (social) gains from trade at the efficient quantity in three different situations. First, the gains from trade averaged over all types of buyer and seller are denoted GFT. Second, let GS1 denote the gains from trade when the worst possible type[1] of seller occurs, averaged over the buyer types. Finally, let GB0 denote the gains from trade arising when the worst type of buyer occurs, averaged over seller types. The efficient quantity can be implemented if and only if $\mathrm{GS1} + \mathrm{GB0} \geqslant \mathrm{GFT}$ (Theorem 1).

This leads to a significant insight not readily accessible in the discrete quantity model. GS1 (GB0) is the largest lump sum payment that can be extracted from the seller (buyer) independently of type, given that the seller (buyer) obtains the full gains from trade. Thus, whenever the efficient quantity can be implemented, the following kind of mechanism will work. Charge the seller (buyer) the amount GS1 (GB0) as a participation or entry free. Then offer the complete gains from trade GFT to both buyer and seller (i.e., the seller sells at the buyer's value and the buyer buys at the seller's cost). This makes honest reporting a dominant strategy, and the mechanism loses exactly the gains from trade GFT (GFT is collected once and paid out twice on average). This is an incentive compatible, individually rational mechanism with net revenue of $\mathrm{GS1} + \mathrm{GB0} - \mathrm{GFT}$. Therefore, it is an implementation of the efficient quantity whenever an implementation exists.[2]

We can now observe that, as long as the types are not too different, there will exist an implementation of the efficient quantity. This requires that trade occur when there is a high cost seller and high value buyer, and when there is a low cost seller and low cost buyer—i.e., even the worst type of one agent will trade with the best type of the other. This insight was not available in the indivisible quantity model at all, except in its extreme form: if one type did not vary at all (degenerate support).

The second variation considered is even more like the "textbook" model, in the sense that the *willingness to pay* for an incremental unit is identified with the *opportunity cost* of giving up a unit. Thus, it is endogeneously determined whether an agent is a buyer or a seller, depending on price and the quantity of the good already in the agent's possession.

[1] The worst type will be unambiguous in the model: higher cost at every quantity.

[2] This mechanism requires a banker, or budget breaker.

I the model I develop, the *hidden endowments model*, an agent witl quasilinear preferences knows his own endowment, but not the other agents endowments, of a good. Ex ante, before the endowments are realized, th agents are symmetric. I will characterize the class of preferences in whicl efficient allocations are feasible[3] for every endowment distribution.

This shows that, when agents are sufficiently ex ante symmetric, th inefficiency found by Myerson and Satterthwaite vanishes. In som environments, symmetry is unrealistic. For example, it is unlikely that a firm will sell labor to a labor union. On the other hand, there are som environments where symmetry is realistic. For example, negotiations o trade matters between countries may be reasonably modelled as symmetric On a more mundane level, stamp collectors are often both buyers an sellers, depending on their preference for the particular stamp in question.

One other example of interest is the credit cooperative, or rotating saving and credit association, used in underdeveloped nations.[4] Individuals (usuall subsistence farmers) in a rural area, without access to other sources of funds form a cooperative to loan each other money, depending on their curren demand and supply. These individuals are more or less ex ante symmetric and receive information about their current situation and value of funds ver so often (whether such information is private is another matter). The credi cooperative attempts to implement the efficient allocation of borrowing an lending. One may also argue that a credit union in a developed country face a similar environment and objective. I am hesistant to take these application too seriously, since the assumption of quasilinear preferences is unlikely to b satisfied in any real world application.

The related literature, on implementing allocations which are efficient o maximize a welfare criterion, divides in a natural way into two classes. In one class, no individual rationality assumption is imposed. This class studies an environment where agents can be coerced into participating (see, for example, Groves and Ledyard [7], D'Aspremont and Gerard–Vare [4], or, more recently, Palfrey and Srivastava [15] and the reference therein). This paper is a member of the second class, which impose a interim individual rationality constraint. Each agent must willingly agree t participate in the game induced by the mechanism after he learns his own private information or type. Auction papers fall in this class, for it i assumed the agent knows his own valuation of the object for sale before h decides to attend the auction mechanism, and must anticipate nonnegativ

[3] The mechanism must satisfy individuality for every realization of the endowment, fo without individual rationality, the result is a special case of d'Aspremont and Gerard–Vare [4].

[4] See, for example, Ghatak [5] and Von Pischke, Adams, and Donald [18].

expected rents from the auction for every possible valuation he might possess.

Since Myerson and Satterthwaite's theorem, the literature in the second class has primarily explored mechanisms which are efficient in some sense other than ex post. Myerson and Satterthwaite [14] identified the ex ante efficient allocations. Wilson [20] demonstrates the ex ante efficiency, and convergence to ex post efficiency as the number of traders diverges, of the double auction, and his results were recently extended by Gresik and Satterthwaite [6]. Several authors have analyzed ex ante efficient mechanisms when time and discounting have been introduced (e.g., Ausubel and Deneckere [2]). Linhart, Radner, and Satterthwaite [11] provide a nice overview of these developments. Spulber [19] considers interim efficient mechanisms.

With two important exceptions, the literature subsequent to Myerson and Satterthwaite shares a set of assumptions which includes the assumption that there is an ex ante identified buyer and seller; that is, either the seller sells the buyer a single unit, or no trade occurs. In Cramton, Gibbons, and Klemperer [3] and in Spulber [19], this assumption is relaxed. In the continuous quantities model of the present study, trade only flows one way but the efficient quantity of trade depends on realization of private information. In Spulber, positive quantities are optimal by assumption. In the hidden endowments model and in Cramton, Gibbons, and Klemperer, any agent may be a buyer, depending on the realization of types. In all of these models, ex post efficient trade may be possible. The interpretation of this development will be explored more fully in the conclusion, but these papers receive the interpretation that ex ante symmetry permits efficient exchange, even in the presence of interim asymmetries.

With agents that are neutral to monetary risks, it matters generally whether the mechanism can serve as a budget-breaker or banker which breaks even on average but may earn money or sustain losses in any particular realization (i.e., ex ante versus ex post budget balance). I shall presume the existence of a banker in calculating the necessary and sufficient conditions for implementing ex post efficiency. However, it should be noted that if a banker is necessary, the two trading agents cannot implement the efficient solution by themselves. In the hidden endowments model, a banker is unnecessary in the event that ex post efficient trade is possible. In the continuous quantities buyer–seller model, the necessity of a banker is not known.

The next section offers a generalization of Myerson and Satterthwaite bilateral trade model. The subsequent section develops and analyzes the hidden endowments model. The final section offers conclusions.

CONTINUOUS QUANTITIES

Let s be the seller's private information, and t the buyer's private infor mation. As in Myerson–Satterthwaite, both parties are presumed to be risl neutral in money. The seller has cost $c(q, s)$ of quantity q, while the buye values quantity q at $v(q, t)$. It is assumed that s and t are independently dis tributed with continuous distribution functions, and thus I may presum that s and t are uniformly distributed on $[0, 1]$ without loss of generality. I assume that c is convex nondecreasing in q and v is strictly concave anc increasing in q, and both are twice continuously differentiable. Only non negative quantities q are allowed. I further assume

$$(\forall s)\ c(0, s) = 0, \qquad (1$$

$$(\forall t)\ v(0, t) = 0, \qquad (2$$

$$(\forall q > 0)(\forall s)\ c_{qs}(q, s) \geqslant 0, \qquad (3$$

$$(\forall q) > 0)(\forall t)\ v_{qt}(q, t) \geqslant 0, \qquad (4$$

$$(\exists q_0)\ v_q(q_0, 1) - c_q(q_0, 0) < 0. \qquad (5$$

Subscripts are used to denote partial derivatives. Define

$$q^*(s, t) \equiv \arg\max_{0 \leqslant q} v(q, t) - c(q, s). \qquad (6$$

Thus, for $q^*(s, t) > 0$,

$$v_q(q^*(s, t), t) = c_q(q^*(s, t), s), \qquad (7$$

$$q_s^*(s, t) \leqslant 0, \qquad (8$$

and

$$q_t^*(s, t) \geqslant 0. \qquad (9$$

Define

$$G(s, t) \equiv v(q^*(s, t), t) - c(q^*(s, t), s)$$

Conditions (1) and (2) establish that the no trade utility levels are zero Condition (3) requires that an increase in s increase marginal cost at all quantity levels, and thus a "high s" seller is unambiguously a high cost

[5] To see this, suppose s has cumulative distribution function F. Then $F(s)$ is uniformly dis- tributed. Thus, letting the private information be $F(s)$ establishes the claim. If F is constant on an interval, some types may be mapped together, but this arises only if the types that are not distinguished have probability zero of occurring.

seller. Similarly, (4) makes the buyer's marginal valuation increasing in t, and thus high type buyers have unambiguously greater demand. The strict concavity of v, combined with the convexity of c, guarantees the maximum in (6) is unique, and (5) insures q^* is finite. Note that negative quantities are not allowed.[6] The differentiability assumptions insure that q^* is differentiable if nonzero and q^* nonincreasing in s and nondecreasing in t, by (3) and (4), respectively. G is the gains from traded associated with any realization (s, t). I use the following notational conventions. All integrals range over crossproducts of $[0, 1]$, whose dimension is indicated by the differential. The arguments (s, t) of q^* will be suppressed unless clarity would suffer. All proofs are contained in the appendix.

By the revelation principle (Harris and Raviv [10], Myerson [13]), attention may be restricted to mechanisms for arranging trades as follows. Both parties report their signals to the mechanism, which then dictates monetary transfers and quantities exchanged. The agents find it optimal to report their signals honestly (incentive compatibility). By assumption, the agents expect nonnegative rents from participation (interim individual rationality). The monetary transfers made need not sum to zero for every (s, t) pair (that is, the mechanism acts as a budget breaker or risk neutral banker), but the mechanism must at least break even in expectation. I restrict attention to mechanisms which are efficient; that is, the quantity exchanged is $q^*(s, t)$. Thus, implementability of the efficient quantity reduces to finding a mechanism satisfying incentive compatibility and individual rationality, and which provides a nonpositive average transfer.[7] The following theorem characterizes the net transfer of any mechanism implementing the quantity q^*.

THEOREM 1. *The minimum expected transfer of any mechanism implementing the efficient quantity q^* is*

$$\Phi = \iint G(s, t)\, ds\, dt - \int G(s, 0)\, ds - \int G(1, t)\, dt. \tag{10}$$

Theorem 1 receives the following interpretation. Any mechanism which implements the efficient quantity provides both buyer and seller with the

[6] It is this lack of symmetry that motivated the analysis of the hidden endowments model in the next section.

[7] Spulber [19] has a related result for a special case of this model, which shows that efficiency can be obtained if the gains from trade exceed the information rents, or expected profits, of the agents. Spulber does not observe that the information rents equal the whole surplus minus a constant, the constant being the expected surplus associated with the worst type. The proof relies heavily on efficiency, via Eq. (7), so that the proof technique is uninformative about implementing other quantity functions.

entire gains from trade, minus any amount that can be charged as a lum[p] sum participation fee that every possible type will pay. This provides ne[t] revenue to the mechanism equal to the gains from trade associated with th[e] worst type of each agent, minus the average gains from trade. If thes[e] participation charges are sufficiently large relative to the average gain[s] from trade, then an efficient mechanism exists, and otherwise not.

COROLLARY 2. *If* $q^*(0, 1) > 0$ *and*

$$(\forall s)\ q^*(s, 0) = 0, \tag{11}$$

or

$$(\forall t)\ q^*(1, t) = 0, \tag{11'}$$

then $\Phi > 0$, *that is, there exists no mechanism inmplementing the efficien[t] quantity that breaks even on average.*

The corollary follows from noting that (11) forces $G(s, 0) = 0$, and tha[t] $G(s, t) \geqslant G(1, t)$ for every t, by (3) and (4), with strict inequality for $(s, t$[)] in a neighborhood of (0, 1). Using (11′) is analogous.

This extends the Myerson–Satterthwaite result to the case of continuou[s] quantities. In the actual Myerson–Satterthwaite result,

$$v(q, t) = \min\{qt, t\}$$

and

$$c(q, s) = \min\{qs, s\}.$$

Because the differentiability assumptions are not satisfied for these func[-]tional forms, the Myerson–Satterthwaite result cannot be taken as a specia[l] case. In particular, the proof relies on the continuous differentiability of q^*[.] The conditions (11) and (11′) play the same role as presuming that th[e] supports of the densities intersect, as in Myerson and Satterthwaite, sinc[e] trade occurs at (0, 1) and not at one of $(1, t)$ or $(s, 0)$. If both (11) an[d] (11′) hold, the level of subsidy required to achieve efficiency is precisely th[e] total gains from trade (the expected value of $v(q^*, t) - c(q^*, s)$). This ha[s] the interpretation that each agent expects to receive the total gains fro[m] trade in order to be induced to honestly reveal his signal.[8]

[8] This parallels a result of Myerson and Satterthwaite: if the supports of the densitie[s] coincide, any mechanism implementing the efficient quantity must subsidize the agents by th[e] total gains from trade. Indeed, each agent individually expects to earn the total gains fro[m] trade, if incentive compatibility and individual rationality are satisfied.

Theorem 1 allows for efficient exchange to be possible, and I provide an example where efficient exchange is implementable. This is not as trivial as it might appear, since the mechanism must implement the efficient q^*, which varies as s and t vary.

EXAMPLE 1.[9] For $a > 1$, let $c(q, s) = sq^2$, and

$$v(q, t) = 2q - (a - t)q^2.$$

It is easily verified that

$$q^*(s, t) = (a + s - t)^{-1}$$

and

$$\Phi = \iint \frac{1 - a}{(a - t + s)^2}\, ds\, dt < O.$$

HIDDEN ENDOWMENTS

We now consider an alternative model, in which the agents are ex ante identical, and it depends on the realization of the privately observed information whether an agent is to be a buyer or seller. There are n agents who are risk neutral to monetary gambles. For simplicity, I presume the private information is the agent's endowment of a consumption good. Denote the endowment of agent i by x_i. I presume that each agent's preferences are identical and quasilinear in the consumption good and money. The preferences can be represented by a strictly concave increasing function u, which is the monetary value of the consumption good. That is, the agent's utility is $u(c) + t$, where c is his consumption of the good and t is the transfer of money he receives. u is presumed to have a continuous fourth derivative. $x_1, ..., x_n$ are identically and independently distributed random variables with a distribution function $F(x_i)$, which possesses a continuous density, and I let the support of F', denoted Ω, be contained in $[0, 1]$.

Efficiency of a mechanism in this environment reduces to equal sharing of the consumption good x, since this equates marginal rates of substitution. Thus, efficiency is characterized by providing each agent with a quantity q_i^* satisfying

$$q_i^* = \frac{1}{n} \sum_{j=1}^{n} x_j - x_i. \tag{12}$$

[9] Spulber [19] provides an example as well.

The following notation is convenient:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i,$$

$$\bar{X}_{-i} = \frac{1}{n} \sum_{j \neq i} x_j,$$

$$dF(\bar{X}) = \prod_{i=1}^{n} dF(x_i),$$

$$dF(\bar{X}_{-i}) = \prod_{j \neq i} dF(x_j),$$

$$\bar{U} = \int u(\bar{X})\, dF(\bar{X}). \qquad (13$$

Theorem 3 characterizes the conditions under which an efficient solutio to this problem is possible. Note that, in an efficient solution, the agent ha the ability to lie at least locally without threat of detection, since the larges amount of his endowment x_i he will ever be asked to give up i $((n-1)/n)x_i$. This allows local deviations from honesty without any issu of verifiability (as would arise if the mechanism required the agent t supply more of the good than he possessed). As it turns out, the desir to be honest locally around the agent's true endowment characterize incentive compatibility globally.

THEOREM 3. *There exists an implementation, with an ex ante balance budget, of the efficient solution* (12) *if and only if*[10]

$$(\forall y \in \Omega) \int u\left(\bar{X}_{-i} + \frac{y}{n}\right) dF(\bar{X}_{-i}) \geqslant \frac{1}{n} u(y) + \frac{n-1}{n} \bar{U}. \qquad (14$$

If so, there is an implementation with an ex post balanced budget.

Remark. The "if" part of this theorem, in the case of two agents, i remarkably easy to demonstrate. Define

$$s(y) = \int u(\tfrac{1}{2}(y + x))\, dF(x),$$

which is the expected direct consumption utility for an agent possessing y units of the good after trade. The transfer that works has agent 1 pay agen

[10] In proving this result, we shall presume F has a continuous density, to establish the "onl if" part of the proposition. This is the only time this assumption is invoked.

the amount $s(x_2)-s(x_1)$ for $\frac{1}{2}(x_2-x_1)$ units of the good, when (x_1, x_2) re the reported endowments, and analogously, agent 2 pays agent 1 the mount $s(x_1)-s(x_2)$ for $\frac{1}{2}(x_1-x_2)$ units. Agent 1's expected utility, if his ndowment is x_1 and he reports y, is

$$\pi(x_1, y)=\int [u(x_1+\tfrac{1}{2}(x_2-y))+s(y)-s(x_2)]\, dF(x_2).$$

It is trivially verified that π is maximized at $y=x_1$ (see Appendix for roof). Thus, individual rationality reduces to

$$2s(x_1)-\int s(x_2)\, dF(x_2)\geq u(x_1)$$

vhich coincides with (14).

The condition (14) has an interpretation as a statement about preferenes over lotteries. Let $x_1, \ldots, x_n$ be identically and independently distributed andom variables. Consider first the random variable $Z(y)$ which is given y

$$Z(y)=\frac{1}{n}\left(y+\sum_{i=1}^{n-1} x_i\right),$$

nd the second random variable $Y(y)$ which takes on the value y with robability $1/n$ and takes on the value $\sum_{i=1}^{n} x_i/n$ with probability $(n-1)/n$). The condition (14) requires that, knowing y, the agent prefers he random variable Z over the random variable Y. Several observations an be made. Let μ, σ^2 be the mean and variance of x_i, respectively. Then

$$EZ(y)=EY(y)=\frac{1}{n}[y+(n-1)\mu], \tag{15}$$

$$\mathrm{VAR}(Z(y))=\frac{n-1}{n^2}\sigma^2, \tag{16}$$

$$\mathrm{VAR}(Y(y))=\frac{n-1}{n^2}[\sigma^2+(y-\mu)^2]. \tag{17}$$

Thus, for any value of y, Z represents the same mean and a lower ariance than Y. It is not generally the case, however, that Z dominates Y n the second order stochastic dominance sense, which is equivalent to very risk averse agent preferring Z to Y, which would in turn ensure that (14) held. Indeed, the situation is somewhat more extreme than this. In the ext pair of results, I show that Z second-order stochastically dominates Y f and only if F is a binomial distribution.

THEOREM 4. *For every distribution F with support Ω containing at lea three points, and for every $n \geqslant 2$, there exists a concave u such that* (14) *fa for some $y \in \Omega$.*

Theorem 4 required at least three values so that there would be "middle type." Intuitively, this comes about so that some type has tl option of either understating or overstating his endowment, and it precisely this type that has an incentive not to participate (for some utili function). However, if there is no middle type, risk aversion is sufficient guarantee implementability for binomial random variables, as the followi rather trivial result shows.

LEMMA 5. (14) *holds for all two point distributions when $n=2$ if ar only if u is concave.*

The strategy of placing assumptions on the distribution of types t guarantee implementability for any concave u does not result in a sati factory theory. That is, no matter what the distribution of endowmen (as long as three different endowments are possible), there will be som specification of preferences that prevents implementation.[11] The strategy o restricting the class of preferences, however, is a more fruitful path. Ou first result in this direction is that, no matter what the distribution o endowments, and for all n, if u displays constant absolute risk aversion efficient implementation is possible.[12] This shows that the class of u satisfying (14) for all distributions is nontrivial. This special case ha remarkably straightforward proof, since the multiple integrals of utilit become products of integrals.

THEOREM 6. *If $u(x) = -e^{-ax}$, for $a > 0$, then* (14) *holds.*

For the case of two agents, I have a simple condition equivalent to (14 holding. Define δ by:

$$\delta(y) = u'''(y)/u''(y).$$

δ is the "prudence" measure for the precautionary savings demand analyzed by Kimball [9], and the reader is referred there for an interpreta tion of δ, which parallels the standard treatment of risk aversion measures Intuitively, δ matters when agents can take ex ante actions to reduce risks

[11] Indeed, if $u(x) = \min\{x, 0\}$ and x is normally distributed with mean 0, then (14) fail (*at* $y=0$) for every $n \geqslant 2$. Thus, even fixing F and u, there cannot be a theorem which state that (14) holds for n sufficiently large.

[12] Note that the set of functions satisfying (14) for any F also includes the less interestin class of Tobin utility functions: $u(x) = ax - bx^2$, for $b \geqslant 0$, by (15)–(17).

ɔ that first derivatives of utility determine risk premia. The following ·mma is needed to establish our result for the two agent case.

LEMMA 7. *Suppose u is concave. Then δ is nondecreasing if and only if, ɔr all distributions F,*

$$u'(y) = \int u'(x)\, dF(x) \Rightarrow u''(y) \geq \int u''(x)\, dF(x).$$

This allows us to prove the main result that, if δ is nondecreasing, then ıere exists an efficient implementation for the two person ex ante ymmetric bargaining problem.

THEOREM 8. *Suppose $n=2$ and u is strictly concave. Then δ is non-ecreasing if and only if* (14) *holds for all distributions F, that is, there xists an efficient solution to the bargaining problem for all endowment istributions.*

It is worth noting that constant absolute and constant relative risk version satisfy δ nondecreasing.

For the two agent case, it is possible to implement ex ante efficient xchange for a reasonably large class of hidden endowment models. The ⁄Iyerson–Satterthwaite result depends heavily on the preferences (which, ıy virtue of the discrete commodity, embody satiation) and the trans-ctions technology (which generally prohibits equality of marginal rates of ubstitution). The hidden endowments model alters these assumptions to ıe closer to the textbook demand model, wherein efficiency is determined ıy equality of marginal rates of substitution. Thus, ex ante there is no buyer," and the realization of endownments or preferences will determine ⁄ho buys and who sells.

It should be noted that x_i need not be interpreted as an endowment, but ıay merely represent a type, with utility of consumption (q, t) represented ıy $u(x+q)+t$, where t is the monetary transfer. The difference from the terature, in this case, is that negative quantities are permitted, and an gent who does not wish to purchase the good at a given price may wish ɔ supply it.

We see that ex post efficiency is not as unlikely as the Myerson–atterthwaite theorem would lead one to expect.

CONCLUSION

The inability of agents to arrange efficient trades in a world with private ıformation is disturbing. In particular, any mechanism will leave the agents with a desire to renegotiate the outcome after the trades dictate by the mechanism have been made, since generally a suboptimal level (trading will have occurred. This paper characterizes the class of enviror ments where implementation is feasible with continuous quantities in tw models. Implementation is generally feasible with correlated signals, as i demonstrated in McAfee and Reny [12], under a hazard rate assumptior

This paper reinforces the conclusion of Cramton, Gibbons, and Klem perer [3], which adopts the Myerson–Satterthwaite framework except tha the initial ownership of the good may be more symmetrically distributec They show that if the initial (ex ante) distribution is sufficiently symmetric efficient allocation is possible. In the same way, the hidden endowment model presented in this paper symmetrizes the agents, in that any agen may be a buyer, depending on the realization of private information. Fo a large class of preferences, ex post efficient allocation is possible.

While this theme, that ex ante asymmetries rather than informationa asymmetries prohibit efficient allocation, is less clear in the first model, i is still present. Although the mechanism can not threaten the buyer witl providing the good in the efficient allocation, it can threaten him witl receiving less of the good, smoothing out the exchange relative t(Myerson–Satterthwaite, and reducing the impact of the ex ante asymmetry

We can view the hidden endowments model as a special case of th(continuous quantities model with a slightly different constraint. Conside the case of two agents, and define

$$v(q, t) = u(q + 1 - t) - u(1 - t)$$

and

$$c(q, s) = u(1 - s) - u(1 - s - q).$$

Here u is the hidden endowments utility function. This particular functiona form satisfies the assumptions of the continuous quantities model, and ye is in the hidden endowments framework (here $x_1 = 1 - t$, $x_2 = 1 - s$, th(reservation utility has been embedded in c and v functions, and $q^*(s, t) = \frac{1}{2}(t - s)$, or $\max\{0, \frac{1}{2}(t - s)\}$ if trade is permitted to flow one way). Thus, w(can see that efficiency in this special case is not possible if trade only flow: one way ($q \geqslant 0$),[13] and may be possible, depending on u, if trade is allowec to flow either way. That is, the inefficiency does not spring from asym metric information, but asymmetric information in conjunction with aı *ex ante* asymmetry (that q is restricted to be nonnegative). Thus, the maiı conclusion of this paper is that *ex ante* asymmetries (that one agent i: the buyer and one is the seller), which are not asymmetric information

[13] Since $q^*(s, 0) = q^*(1, t) = 0$. Corollary 2 demonstrates this fact.

ınd not interim asymmetries (the buyer's and seller's valuations), which ıre asymmetric information, prevent efficient trades in the Myerson–Satterthwaite model.

APPENDIX

The following observation is used several times throughout the appendix. believe it was initially obtained by Guesneries and Laffont [8], in a more general version.

LEMMA 0. *Suppose an agent of type t who reports r receives profits of $\pi(r, t)$ and $(\partial\pi/\partial r)(t, t)=0$, and $(\partial^2\pi/\partial r\,\partial t)(r, t)\geqslant 0$. Then π is maximized over r at $r=t$.*

Proof. $(\partial\pi/\partial r)(r, t)\gtreqless 0$ as $t\gtreqless r$, and thus π is maximized at $r=t$. ∎

THEOREM 1. *The minimum expected transfer of any mechanism implementing the efficient quantity q^* is*

$$\Phi=\iint G(s, t)\,ds\,dt-\int G(s, 0)\,ds-\int G(1, t)\,dt. \tag{10}$$

Proof. Consider a buyer with signal t who reports r. His return is

$$u(t)=\max_r \int v(q^*(s, r), t)\,ds-p(r), \tag{A1}$$

vhere $p(r)$ is his expected payment to the mechanism when he reports r. ncentive compatibility, with the envelope theorem, implies[14]

$$u'(t)=\int v_t(q^*(s, t), t)\,ds. \tag{A2}$$

[14] The differentiability of u is proved as follows. Incentive compatibility yields directly

$$\frac{\int_0^1 v(q^*(s, t), t)-v(q^*(s, r), t)\,ds}{t-r}\geqslant\frac{p(t)-p(r)}{t-r}$$
$$\geqslant\frac{\int_0^1 v(q^*(s, t), r)-v(q^*(s, r), r)\,ds}{t-r}.$$

'hus differentiability of p reduces to the differentiability in r of $\int_0^1 v(q^*(s, r), t)\,ds$. Since v and are assumed twice continuously differentiable q^* is continuously differentiable except around he single point s_0 such that $q^*(s_0, r)=0$ and, for $s<s_0$, $q^*(s, r)>0$. Thus, p is continuously differentiable. The argument shows u is continuously differentiable.

The expected payment to the mechanism is (using (A1), (A2) and integra-tion by parts)

$$Ep = \int p(t)\,dt$$

$$= \iint v(q^*, t)\,ds - u(t)\,dt$$

$$= \iint v(q^*, t) - (1-t)\,v_t(q^*, t)\,ds\,dt - u(0). \qquad \text{(A3}$$

Since $u'(t) \geqslant 0$ (by integrating (4) over q and noting that $v_t(0, t) = 0$ b (2)), individual rationality is equivalent to

$$u(0) \geqslant 0. \qquad \text{(A4}$$

To establish the incentive compatibility of the mechanism in (A1) ar (A2), note that by (4) and (9)

$$\frac{\partial^2}{\partial r\,\partial t} \int v(q^*(s, r), t)\,ds - p(r) = \int v_{qt}(q^*(s, r), t)\,q_t^*(s, r)\,ds \geqslant 0. \qquad \text{(A5}$$

Thus, viewing u in (A1) as a function of r and t (not taking the maximu in (A1)), we have

$$u_r(t, t) = 0, \qquad \text{(A6}$$

and

$$u_{rt}(r, t) \geqslant 0. \qquad \text{(A7}$$

(A6) follows from (A2), and (A5) implies (A7). Lemma 0 implies that u maximized at $r = t$, as desired.

Now suppose the mechanism pays the seller who reports r an amou $w(r)$ on average. The seller expects

$$\pi(s) = \max_r\, w(r) - \int c(q^*(r, t), s)\,dt. \qquad \text{(A8}$$

Thus,

$$\pi'(s) = -\int c_s(q^*(s, t), s)\,dt. \qquad \text{(A9}$$

rom (A8) and (A9), the expected payment made to the seller is

$$Ew = \int w(s)\, ds$$

$$= \pi(1) + \iint c(q^*, s) + sc_s(q^*, s)\, dt\, ds. \tag{A10}$$

ince $\pi'(s) \leqslant 0$, from (1), (3), and (A9), individual rationality reduces to

$$\pi(1) \geqslant 0. \tag{A11}$$

Incentive compatibility for the seller is established analogously to A3)–(A5), using Lemma 0, and noting (8). The net transfer made by the echanism is

$$\Phi = Ew - Ep. \tag{A12}$$

learly, the minimum transfer occurs when $u(0) = \pi(1) = 0$. By (7)

$$\frac{d}{dt}(1-t)\, v(q^*, t) = -v + (1-t)\, v_t + (1-t)\, v_q q_t^*$$

$$= -v + (1-t)\, v_t + (1-t)\, c_q q_t^*,$$

$$\frac{d}{ds} sc(q^*, s) = c + sc_s + sc_q q_s^*$$

$$= c + sc_s + sv_q q_s^*,$$

henever q^* is differentiable (almost everywhere). Thus, integrating by arts,

$$\int -v + (1-t) v_t\, dt = -v(q^*(s, 0), 0) - \int (1-t)\, c_q q_t^*\, dt$$

$$= -v(q^*(s, 0), 0) + c(q^*(s, 0), s) - \int c(q^*, s)\, dt,$$

$$\int c + sc_s\, ds = c(q^*(1, t), 1) - \int sv_q q_s^*\, ds$$

$$= c(q^*(1, t), 1) - v(q^*(1, t), t) + \int v(q^*, t)\, ds,$$

here v, c are evaluated at (q^*, t) and (q^*, s), respectively, and q^* is evaluated at (s, t) unless otherwise indicated. By (A3), (A4), (A10), (A11 (A12), and Fubini's theorem and integration by parts,

$$\Phi = \iint -v + (1-t)v_t \, dt \, ds + \iint c + sc_s \, ds \, dt$$

$$= \int c(q^*(s, 0), s) - v(q^*(s, 0), 0) - \int c(q^*, s) \, dt \, ds$$

$$+ \int c(q^*(1, t), 1) - v(q^*(1, t), t) + \int v(q^*, t) \, ds \, dt$$

$$= \iint G(s, t) \, ds - \int G(s, 0) \, ds - \int G(1, t) \, dt. \quad \blacksquare$$

THEOREM 3. *There exists an implementation, with ex ante budg balance, of the efficient solution* (13) *if and only if*

$$(\forall y \in \Omega) \int u\left(\bar{X}_{-i} + \frac{y}{n}\right) dF(\bar{X}_{-i}) \geqslant \frac{1}{n} u(y) + \frac{n-1}{n} \bar{U}. \qquad (14$$

If so, there is an implementation with an ex post balanced budget.

Proof. (if) When (14) holds, there is a simple mechanism whic implements the efficient solution. The mechanism takes reports $x_1, ..., x_n$ (endowments and assigns the efficient quantity to each agent. Define

$$s(y) = \int u(\bar{X}_{-i} + y/n) \, dF(\bar{X}_{-i}).$$

Agent i is asked to pay

$$p_i(x_i, ..., x_n) = \sum_{j \neq i} s(x_j) - (n-1) \, s(x_i).$$

Note that

$$\sum_{i=1}^{n} p_i(x_i, ..., x_n) = 0.$$

That is, no budget breaker is required. Note as well that agent i's expecte payment, conditioned on a report r, is

$$\psi(r) = (n-1)\left[\bar{U} - \int u(\bar{X}_{-i} + r/n) \, dF(\bar{X}_{-i})\right].$$

the agent's endowment is x_i and he reports an endowment of r, he xpects net utility (over the reservation value $u(x_i)$) of

$$\pi = \int u\left(x_i + \bar{X}_{-i} - \frac{n-1}{n} r\right) dF(\bar{X}_{-i}) - \psi(r) - u(x_i)$$

$$\frac{\partial \pi}{\partial r} = -\frac{n-1}{n} \int \left[u'\left(x_i + \bar{X}_{-i} - \frac{n-1}{n} r\right) - u'(\bar{X}_{-i} + r/n)\right] dF(\bar{X}_{-i})$$

$$\frac{\partial^2 \pi}{\partial x_i\, \partial r} = -\frac{n-1}{n} \int u''\left(x_i + \bar{X}_{-i} - \frac{n-1}{n} r\right) dF(\bar{X}_{-i}) > 0.$$

'hus, π is pseudoconcave in r and by Lemma 0 is maximized when $\pi/\partial r = 0$, which occurs at $r = x_i$. Given that the agent reports honestly, his rofits are

$$\pi = n \int u(\bar{X})\, dF(\bar{X}_{-i}) - u(x_i) - (n-1)\, \bar{U}.$$

'hus, individual rationality holds if (14) holds, that is, $\pi(x_i) \geqslant 0$.

(only if): Consider a mechanism which, without loss of generality, harges the agent $p(r)$ when he reports an endowment r. If the mechanism s efficient, this agent, if his true endowment is x_i, expects rents (net of (x_i) obtained by not participating) of

$$\pi = \int u\left(x_i + \bar{X}_{-i} - \frac{n-1}{n} r\right) dF(\bar{X}_{-i}) - u(x_i) - p(r). \tag{A13}$$

ncentive compatibility forces

$$p'(x_i) = -\frac{n-1}{n} \int u'(\bar{X})\, dF(\bar{X}_{-i}). \tag{A14}$$

ntegrating (A14) by parts, and using Fubini's theorem, the net payments o the mechanism are

$$\begin{aligned} 0 \leqslant Ep &= \int p(x_i)\, dF(x_i) \\ &= p(0) + \int (1 - F(x_i))\, p'(x_i)\, dx_i \\ &= p(0) - \frac{n-1}{n} \iint (1 - F(x_i))\, u'(\bar{X})\, dx_i\, dF(\bar{X}_{-i}) \\ &= p(0) + (n-1) \int u(\bar{X}_{-i})\, dF(\bar{X}_{-i}) - (n-1)\, \bar{U}, \end{aligned} \tag{A15}$$

where $\bar{U}$ is given in (13). Integrating (A14) directly, and using (A15), yield

$$p(y) = p(0) + \int_0^y p'(x_i)\,dx_i$$
$$= p(0) - (n-1)\int u\left(\bar{X}_{-i} + \frac{y}{n}\right) - u(\bar{X}_{-i})\,dF(\bar{X}_{-i}). \qquad \text{(A16}$$

Combining (A13), (A15), and (A16) gives

$$\pi(y) \leqslant n \int u\left(\bar{X}_{-i} + \frac{y}{n}\right) dF(\bar{X}_{-i}) - u(y) - (n-1)\bar{U}. \qquad \text{(A17}$$

Thus, (A17) with individual rationality,

$$(\forall y) \qquad \pi(y) \geqslant 0,$$

implies that (14) holds. ∎

THEOREM 4. *For every distribution F with support Ω containing at lea three points, and for every $n \geqslant 2$, there exists a concave u such that* (14) *fai for some $y \in \Omega$.*

Proof. Let $F_n(z)$ be the probability that $(1/n)\sum_{i=1}^n x_i \leqslant z$. Then th distribution function of $Z(y)$ is

$$H(x) = F_{n-1}\left(\frac{nx - y}{n-1}\right).$$

Similarly, the distribution function of $Y(y)$ is

$$G(x) = \begin{cases} \dfrac{n-1}{n} F_n(x) & \text{if } x < y \\[2ex] \dfrac{n-1}{n} F_n(x) + 1/n & \text{if } x \geqslant y. \end{cases}$$

It is sufficient to prove that H does not dominate G, in the sense of secon order stochastic dominance (see Rothschild and Stiglitz [16]), that i there exists a y_0 such that

$$\int_0^{y_0} G(x) - H(x)\,dx < 0.$$

This is equivalent since the means associated with Z and Y coincide. Choose $y_0 = y$. Then

$$\int_0^y G(x) - H(x)\,dx = \int_0^y \frac{n-1}{n} F_n(x) - F_{n-1}\left(\frac{nx-y}{n-1}\right) dx$$
$$= \frac{n-1}{n}\left[\int_0^y F_n(x) - F_{n-1}(x)\,dx\right].$$

Thus, it is sufficient to prove that F_n strictly dominates F_{n-1}, in the second order stochastic dominance sense, to satisfy the inequality. Let v be any concave function. Define

$$w_j = \frac{1}{n-1} \sum_{i \neq j} x_i.$$

Then, since x_i are iid,

$$Ev\left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) = E\sum_{j=1}^{n} v(w_j)/n$$
$$\geqslant Ev\left(\sum_{j=1}^{n} w_j/n\right)$$
$$= Ev\left(\sum_{j=1}^{n} x_i/n\right).$$

The inequality holds for every realization of $x_1, ..., x_n$, since v is concave. The weak inequality stated will be satisfied with strict inequality if y is in the interior of the support of F, requiring at least three values. ∎

LEMMA 5. (14) *holds for all two point distributions when* $n = 2$ *if and only if* u *is concave.*

Proof. Consider a two point distribution on a, b, with probabilities $p, 1-p$, respectively. Let $c = \frac{1}{2}(a+b)$. Then (14) reduces to

$$pu(\tfrac{1}{2}(a+y)) + (1-p)\,u(\tfrac{1}{2}(b+y))$$
$$\geqslant \tfrac{1}{2}u(y) + \tfrac{1}{2}[p^2u(a) + 2p(1-p)\,u(c) + (1-p)^2u(b)]$$

for $y = a, b$. Using $y = a$ and collecting terms yields

$$(1-p)^2u(c) \geqslant \tfrac{1}{2}[(1-p)^2u(a) + (1-p)^2u(b)] \tag{A18}$$

and similarly, for $y = b$,

$$p^2u(c) \geqslant \tfrac{1}{2}[p^2u(a) + p^2u(b)]. \tag{A19}$$

(A18) and (A19) are equivalent to:

$$u(\tfrac{1}{2}(a+b)) \geqslant \tfrac{1}{2}[u(a)+u(b)],$$

which, with continuity, is in turn equivalent to concavity (see Rudir [17, p. 72]). ∎

THEOREM 6. *If* $u(x) = -e^{-ax}$, *for* $a > 0$, *then* (14) *holds.*

Proof. Let

$$A = \int e^{-ax/n} f(x)\,dx > 0,$$

and

$$v(y) = \int u(y/n + \bar{X}_{-i})\,dF(\bar{X}_{-i}) - \frac{1}{n} u(y) - \frac{n-1}{n} \int u(\bar{X})\,dF(\bar{X})$$

$$= -e^{-\alpha y/n} A^{n-1} + e^{-\alpha y}/n + \frac{n-1}{n} A^n.$$

(14) is equivalent to $(\forall y)\ v(y) \geqslant 0$. Note that

$$v'(y) = \frac{\alpha}{n} [e^{-\alpha y/n} A^{n-1} - e^{-\alpha y}]$$

$$v''(y) = \frac{\alpha^2}{n^2} [-e^{-\alpha y/n} A^{n-1} + ne^{-\alpha y}]. \qquad \text{(A20}$$

Thus $v'(y) = 0 \Rightarrow v''(y) > 0$, and any extreme point is a minimum. Fror (A20) any extreme point satisfies

$$A = e^{-\alpha y/n}.$$

This yields

$$v(y) \geqslant \min_y v(y) = -A^n + \frac{1}{n} A^n + \frac{n-1}{n} A^n = 0. \quad ∎$$

LEMMA 7. *Suppose u is concave. Then δ is nondecreasing if and only i*

$$u'(y) = \int u'(x)\,dF(x) \Rightarrow u''(y) \geqslant \int u''(x)\,dF(x). \qquad \text{(1}$$

Proof of Lemma 7. Let $G(x) = F(u'^{-1}(x))$. Then

$$y = u'^{-1}\left(\int z\,dG(z)\right).$$

Thus, (18) becomes, if $h(z) = u''(u'^{-1}(z))$,

$$h\left(\int z\, dG(z)\right) \geq \int h(z)\, dG(z).$$

This is equivalent to h concave, and thus

$$0 \geq h''(z) = \frac{\delta'(u'^{-1}(z))}{u''(u'^{-1}(z))}.$$

Since $u'' < 0$, we have equivalence of (18) and $\delta' \geq 0$. ∎

THEOREM 8. *Suppose $n = 2$ and u is strictly concave. Then δ is non-decreasing if and only if (14) holds for all distributions F, that is, there exists an efficient solution to the bargaining problem.*

Proof. (if) It is useful to introduce the following notation:

$$\lambda(z, y) = u\,[(\tfrac{1}{2}(z + y)] - \int u[(\tfrac{1}{2}(z + x)]\, dF(x).$$

Then (14) may be expressed as

$$\Phi(y) = \frac{1}{2}\left[\int \lambda(x, y)\, dF(x) - \lambda(y, y)\right] \geq 0. \tag{A21}$$

Let y^* minimize Φ. Then

$$\int u'[(\tfrac{1}{2}(x + y^*)]\, dF(x) - u'(y^*) = 0.$$

This forces

$$\lambda_1(y^*, y^*) = 0. \tag{A22}$$

Note that

$$\lambda_{12}(z, y) = \tfrac{1}{4}u''(\tfrac{1}{2}(z + y)) \leq 0. \tag{A23}$$

By Lemma 7, including the value of $\frac{1}{2}z$ in x and y, we obtain

$$\lambda_1(z, y) = 0 \Rightarrow \lambda_{11}(z, y) \geq 0. \tag{A24}$$

Define $\hat{z}$ by $\lambda_1(\hat{z}(y), y) = 0$. From (A23) and (A24) we have that $\hat{z}$ is non-decreasing. This yields, by (A23),

$$\lambda_1(z, y) \gtreqless 0 \quad \text{as} \quad \hat{z}^{-1}(z) \gtreqless y \quad \text{as} \quad z \gtreqless \hat{z}(y).$$

Thus, in particular, since $\hat{z}(y^*) = y^*$ by (A22), $\lambda(z, y^*)$ is pseudoconvex in z, and takes a global minimum at $z = y^*$. This proves (14) holds:

$$\Phi(y) \geqslant \min_y \Phi(y) = \Phi(y^*) = \tfrac{1}{2}\left[\int \lambda(x, y^*)\, dF(x) - \lambda(y^*, y^*)\right] \geqslant 0.$$

(only if) Suppose there is a y^* satisfying $\delta'(y^*) < 0$. By Lemma 7, there is a distribution F such that

$$u'(y^*) = \int u'(x)\, dF(x) \Rightarrow u''(y^*) < \int u''(x)\, dF(x).$$

At (y^*, y^*), $\lambda_{11} < 0$, and thus $\hat{z}$ is strictly decreasing. The same argument as above shows that $\lambda(z, y^*)$ takes a strict maximum in z at y^*, and thus $\Phi(y^*) < 0$, by (A21). ∎

References

1. A. Alchian and W. Allen, "Exchange and Production: Competition, Coordination and Control," 2nd ed., Wadsworth, Belmont, CA, 1977.
2. L. Ausubel and R. Deneckere, A direct mechanism characterization of sequential bargaining with one sided incomplete information, *J. Econ. Theory* **48** (1989), 18–46.
3. P. Cramton, R. Gibbons, and P. Klemperer, Dissolving a partnership efficiently, *Econometrica* **55** (1987), 615–632.
4. C. d'Aspremont and L. Gerard–Varet, Incentives and incomplete information, *J. Public Econ.* **11** (1979), 25–46.
5. S. Ghatak, "Rural Money Markets in India," MacMillan of India, Delhi, 1976.
6. T. Gresik and M. Satterthwaite, The rate at which a simple market converges to efficiency as the number of traders increases: An asymptotic result for optimal trading mechanisms, *J. Econ. Theory* **48** (1989), 304–332.
7. T. Groves and J. Ledyard, Optimal allocation of public goods: A solution to the "free-rider" problem, *Econometrica* **45** (1977), 783–809.
8. R. Guesnerie and J. J. Laffont, A complete solution to a class of principal–agent problems with an application to the control of a self-managed firm, *J. Public. Econ.* **25** (1984), 329–369.
9. M. Kimball, Precautionary savings in the small and the large, University of Michigan, mimeo, 1988.
10. M. Harris and A. Raviv, Allocation mechanisms and the design of auctions, *Econometrica* **49** (1981), 1477–1499.
11. P. Linhart, R. Radner, and M. Satterthwaite, Introduction: Symposium on non-cooperative bargaining, *J. Econ. Theory* **48** (1989), 1–17.

12. R. P. McAfee and P. Reny, Correlated information and mechanism design, *Econometrica*, forthcoming, 1990.
13. R. Myerson, Optimal coordination mechanisms and generalized principal–agent problems, *J. Math. Econ.* **10** (1982), 67–81.
14. R. Myerson and M. Satterthwaite, Efficient mechanisms for bilateral trading, *J. Econ. Theory* **28** (1983), 265–281.
15. T. Palfrey and S. Srivastava, Mechanism design with incomplete information: A solution to the implementation problem, *J. Polit. Econ.* **97** (1989), 668–691.
16. M. Rothschild and J. Stiglitz, Increasing risk: A definition, *J. Econ. Theory* **2** (1970), 225–243.
17. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1974.
18. J. D. Von Pischke, D. Adams, and G. Donald, Eds., "Rural Financial Markets in Developoing Countries," Johns Hopkins Univ. Press, 1983.
19. D. Spulber, Bargaining and regulation with asymmetric information about demand and supply, *J. Econ. Theory* **44** (1988), 251–268.
20. R. Wilson, Incentive efficiency of double auctions, *Econometrica* **53** (1985), 1101–1115.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium