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A Dominant Strategy Double Auction

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A double auction mechanism that provides dominant strategies for both buyers and sellers is analyzed. This mechanism satisfies the $1/n$ convergence to efficiency of the buyer's bid double auction. In addition, the mechanism always produces full information first best prices; the inefficiency arises because the least valuable profitable trade may be prohibited by the mechanism. The mechanism has an oral implementation utilizing bid and asked prices. *Journal of Economic Literature* Classification Numbers: D44, D62. © 1992 Academic Press, Inc.

INTRODUCTION

The dollar volume of assets exchanged via oral double auctions in the world's stock, bond, and commodity markets exceeds world GNP many times over. For this reason, oral double auctions are perhaps the most important trading mechanisms in the modern economy. In contrast to the voluminous literature on one-sided auctions,[1] little is known about double auctions in general, and oral double auctions in particular.

The lack of knowledge about double auctions arises because the double auctions studied tend to have immensely complicated bidding behavior.[2] For example, Wilson [11] has shown that, if the number of buyers and sellers is sufficiently large, the market satisfies Holmstrom and Myerson's [1] notion of interim efficiency. The stronger notion of ex ante efficiency is obtained only under the peculiar assumption of equal numbers of potential buyers and sellers. Similarly, Satterthwaite and Williams [10] study

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[1] See McAfee and McMillan [4] for a comprehensive survey.

[2] Some of the complexity arises even in the full information case. Suppose there are n buyers and n sellers, all with values drawn from the same distribution. The efficient number of trades is k with probability $\binom{n}{k}^2/\binom{2n}{n}$. The asymmetric information case, of course, inherits the complexity of the full information case.

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the buyer's bid double auction to eliminate strategic behavior on the part of the seller, and thereby simplify the analysis.

In contrast to other studies, the present study examines a mechanism which is remarkably simple to analyze. The simplicity arises out of two aspects of the mechanism. First, both buyers and sellers have dominant strategies, and these strategies involve honest reporting of valuations. Thus, the strategic behavior which is the main complicating factor in other studies is absent from this one. Second, the inefficiency is limited to the loss of at most one efficient trade, and this trade is the least important of all possible efficient trades, as it involves the lowest value buyer and highest cost seller that would be involved in efficient exchange. As in other studies of double auctions, only independent private values will be considered.

One undesirable aspect of the mechanism is that it, on occasion, makes money, although it never loses money. This appears to be necessary to obtain dominant strategies.[3] Moreover, trade on the New York Stock Exchange involves a market specialist who makes money. This connection is discussed in Section 4.

The mechanism can be illustrated with a simpler mechanism that is closely related. Let buyers report their values, and rank them $b_1 \geqslant b_2 \geqslant \cdots$; also rank the sellers' reported costs $s_1 \leqslant s_2 \leqslant \cdots$. Find the efficient trade quantity k (satisfying $b_k \geqslant s_k$ and $b_{k+1} < s_{k+1}$). Let the $k-1$ highest value buyers trade with the $k-1$ lowest cost sellers, with buyers paying b_k and sellers being paid s_k, and the mechanism earning $(k-1)(b_k - s_k)$. This mechanism provides both buyers and sellers with dominant strategies, loses only the least valuable trade, and earns money. The mechanism studied in this paper is related in that it occasionally executes all efficient trades and makes no money, but otherwise charges buyers b_k and pays sellers s_k.

It is important that the money earned by the mechanism be counted as part of the social welfare. Thus, unlike the mechanism of Satterthwaite and Williams, the present study requires an additional, nontrading agent who serves as budget balancer or market maker by soaking up the excess revenue generated by the mechanism. The assumption that a market maker exists is less odious in this environment than in many mechanism design problems because the mechanism never loses money. However, this mechanism could not operate without the additional agent, who does not desire to buy or sell the good.

One important feature of the mechanism is that it can be specified, and equilibrium strategies computed, without knowledge of the distributions that generate valuations. In particular, suppose the distribution F gene-

[3] The results of Hurwicz [2] show that efficient allocation, dominant strategies, and balanced budgets are incompatible. These results, while suggestive, do not directly apply, because the mechanism will implement an ex post inefficient solution.

rating buyers' valuations is not known to the buyers, but is viewed as a random draw from a family of distributions, so that buyers' valuations are all drawn from the same random distribution function. This induces correlation in the buyers' valuations, which would severely complicate the analysis of the buyer's bid double auction, but presents no complication for the present study.[4]

The mechanism I present has a natural oral implementation that is related to the "double dutch" auction studied by McCabe, Rassenti, and Smith [7]. The double dutch auction stops with equality of bid and asked prices; this does not necessarily occur in the auction studied here.

Satterthwaite and Williams [10] have analyzed the asymptotic efficiency of the buyer's bid double auction and shown that buyers underreport their true valuations by an amount of order $1/n$, where n is the number of buyers and sellers. They interpret this to mean that the importance of strategic behavior vanishes quickly as the number of traders increases. However, their result does not directly address the size of the efficiency loss arising from strategic bidding as the number of traders increases, which is the focus of the present analysis. Because of the dominant strategies, no underreporting occurs in equilibrium,[5] and the issue considered by Satterthwaite and Williams does not arise. The present study shows that the expected efficiency loss (for the dominant strategy mechanism, not the buyer's bid double auction) is also on the order of $1/n$. From this, we can automatically conclude that this is true of the mechanism which maximizes ex ante expected welfare.

The second section describes the environment and analyzes the direct implementation of the oral double auction with dominant strategies. The third section demonstrates the convergence result. The fourth section describes and analyzes the oral dominant strategy double auction, after which a conclusion is offered.

The Direct Implementation

There are m buyers and n sellers in the market. Each buyer i has a privately observed value b_i for a single unit of the good, and each seller j has a privately observed value s_j for the single unit she possesses.

A buyer with value b who pays p and receives a unit of the good obtains utility $b-p$; a buyer paying nothing and not receiving the good obtains

[4] I thank an anonymous referee for pointing this out.

[5] To be precise, no underreporting arises when agents play their dominant strategies. There are other equilibria, discussed in Remark 1. I follow Satterthwaite and Williams and assume agents with dominant strategies play their dominant strategy.

zero utility. Similarly, a seller who gives up her unit of the good valued at s and receiving payment p obtains utility $p-s$, otherwise obtaining zero if no trade and payment arise. Buyers and sellers are neutral to risk, and the utility functions, up to the private valuations, are common knowledge.

The direct implementation of the dominant strategy auction is a direct mechanism. Agents report their valuations to the mechanism, which dictates trades and payments as a function of the reports. Prior to making reports, agents know their own valuation, and the mechanism is common knowledge.

Let $b_1, ..., b_m$ be the buyers' reports and $s_1, ..., s_n$ be the sellers' reports. Define the order statistics:

$$b_{(1)} \geqslant b_{(2)} \geqslant \cdots \geqslant b_{(m)} \tag{1}$$

and

$$s_{(1)} \leqslant s_{(2)} \leqslant \cdots \leqslant s_{(n)}. \tag{2}$$

Note the reverse ordering for buyers and sellers. We shall use the notation (i) for the ith highest valuation buyer or ith lowest cost seller.[6]

I follow the convention

$$b_{(m+1)} = \sup\{b: F(b) = 0\}, \tag{3}$$

$$s_{(n+1)} = \inf\{s: G(s) = 1\}. \tag{4}$$

That is, the fictitious order statistic $b_{(m+1)}$ is the lowest possible value and, similarly, $s_{(n+1)}$ is the highest possible cost. The efficient number of trades is the number $k \leqslant \min\{m, n\}$ satisfying

$$b_{(k)} \geqslant s_{(k)}, \tag{5}$$

and

$$b_{(k+1)} < s_{(k+1)}, \tag{6}$$

where the mechanism dictates that trade occurs if it produces exactly zero surplus. Finally, define

$$p_0 = \tfrac{1}{2}(b_{(k+1)} + s_{(k+1)}), \tag{7}$$

where k is the efficient number of trades, satisfying inequalities (5) and (6). We are now in a position to define the mechanism. The mechanism

[6] In the case of ties, $b_{(i)} = b_{(i+1)}$, the ordering is random, with each tied player having an equal probability of being identified as being player (i). The analogous tie-breaking rule applies to sellers.

arranges reported buyer and seller valuations as in (1) and (2) and determines k and p_0.

The mechanism requires a budget balancer to operate. If $p_0 \in [s_{(k)}, b_{(k)}]$, all efficient buyers and sellers (1) through (k) trade at price p_0. If $p_0 \notin [s_{(k)}, b_{(k)}]$, only buyers and sellers (1) through ($k-1$) trade; in this case buyers pay $b_{(k)}$ and sellers receive $s_{(k)}$, and the mechanism, or the budget balancer, keeps $(k-1)(b_{(k)} - s_{(k)})$. It is important that the money paid to the budget balancer be counted in the surplus, and the convergence result depends on this assumption. This mechanism is called the direct implementation of the dominant strategy auction.

THEOREM 1. *Honesty is a dominant strategy for the direct implementation of the dominant strategy auction.*

Proof. Consider first a buyer with valuation b, and let r denote the buyer's proposed report. Order the remaining buyers' reported values (we are not assuming our buyer has the highest value)

$$b_2 \geqslant b_3 \geqslant \cdots \geqslant b_m,$$

and the sellers' costs,

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n.$$

Let k denote the efficient number of trades given that our buyer trades, i.e., $b_k \geqslant s_k$ and $b_{k+1} < s_{k+1}$, and $p_0 = \frac{1}{2}(b_{k+1} + s_{k+1})$. In the case that our buyer trades, he pays

$$\begin{cases} v_k & \text{if } \; p_0 \notin [c_k, v_k] \text{ and } r \geqslant v_k \\ p_0 & \text{if } \; p_0 \in [c_k, v_k] \text{ and } r \geqslant p_0. \end{cases}$$

If $p_0 \notin [s_k, b_k]$, the buyer's utility is either $b - v_k$ or 0, depending on whether $r \geqslant b_k$; the usual Vickrey argument applies. Similarly, if $p_0 \in [s_k, b_k]$, the buyer trades at p_0 unless the reports a value less than p_0, in which case he is excluded from trading.

Now if there is a tie for being (k), and our buyer reports $r = b_k$, the buyer buys with certainty when $p_0 \in [s_k, b_k]$; so the preceding analysis applies. However, if there is a tie involving our buyer and $p_0 \notin [s_k, b_k]$, the mechanism randomizes over whether our buyer is excluded from trading. This includes the case when $b_k = b_{k+1}$, for this implies $p_0 = \frac{1}{2}(b_{k+1} + s_{k+1}) > \frac{1}{2}(b_{k+1} + b_{k+1}) = b_{k+1} = b_k$; that is, $p_0 \notin [s_k, b_k]$. In this case, our buyer receives $b - v_k$ if $r > b_k$, $\theta(b - v_k)$ if $r = b_k$ for some $\theta \in (0, 1)$, and 0 otherwise. It follows immediately that a buyer with $b \neq b_k$ will never report $r = b_k$, and that a buyer with $b = b_k$ cannot do better than reporting $r = b_k$.

It is useful to note that the price paid by any trader is invariant to that trader's report, provided that the agent trades with positive probability; the only way a trader can affect the price is by eliminating himself from trading. Thus, by reporting a value $r \in [s_k, b_k)$, when $p_0 \notin [s_k, b_k]$, a buyer can reduce the price, but the eliminates himself from purchasing by doing so, for he is now buyer (k). Similarly, if $p_0 \in [s_k, b_k]$, a report of $r < p_0$ can lower the price, but also eliminates our buyer from trade, because there are k efficient traders if $r \geqslant s_k$, or $k-1$ if $r < s_k$, and the report $r < p_0$ places the buyer at k or higher. The case of the seller is similar. ∎

Remark 1. Honesty is not a unique equilibrium. Assuming all valuations are nonnegative, consider the strategies where all buyers report 0 and all sellers report a cost in excess of the maximum possible buyer value. Clearly no agent unilaterally profits from deviating, and this comprises a degenerate equilibrium.

There are also variants of this equilibria, at least in the full information case. Let $l < k$, and suppose the $l-1$ highest value buyers report the maximum possible value, and the $l-1$ lowest cost sellers report a cost of zero. Let buyer (l) and seller (l) report honestly, and finally buyers with values less than $b_{(l)}$ report zero, while sellers with costs greater than $s_{(l)}$ report a value exceeding both the maximum possible buyer's value and $2b_{(l)}$. This also comprises an equilibrium, although it depends, of course, on all buyers knowing $b_{(l)}$ and sellers knowing $s_{(l)}$, except in the case where $l=0$, which was the first case considered. Thus, there are equilibria where dominant strategies are not played, because buyers can report anything given that sellers are expensive, and sellers will report high values, given that buyers report very low values.

Even if sellers follow their dominant strategy, there may remain equilibria with no trade. Suppose possible seller costs range from L to H, with $H < 2L$. There is an equilibrium where all buyers report zero valuations. To see this, note that

$$s_{(1)} \geqslant L > \tfrac{1}{2}H \geqslant \tfrac{1}{2}(s_{(2)} + b_{(2)}) = p_0,$$

since $b_{(2)} = 0$. Thus, a unilateral deviation by a buyer will not permit that buyer to trade at any price, since $p_0 \notin [s_{(1)}, b_{(1)}]$.

Consequently, I must assume that agents play their dominant strategies. Honesty is the only dominant strategy, for reports other than one's valuation lead to losses or lost profits in some realizations.

Remark 2. The dominant strategies are unaffected by increasing concave transformations of utility.

THE RATE OF CONVERGENCE TO EFFICIENCY

Satterthwaite and Williams establish that underreporting by buyers in the buyer's bid double auction converges to zero at a rate at least $1/m$, where m is the number of buyers and sellers. That is, if B_m is a symmetric equilibrium bidding function, then there is a continuous function c so that $x - B_m(x) \leqslant c(x)/m$. The function c will generally depend on the distributions F and G from which buyer and seller valuations are drawn, but does not depend on the equilibrium selected, provided it is symmetric and agents with dominant strategies play their dominant strategy. Moreover, if the density associated with buyers' valuations goes to zero at some point, c may diverge at that point, but by assumption this is permitted only at the endpoints of the interval support of the density.

The Satterthwaite and Williams result does not directly provide information about the rate of convergence to efficiency, which is the focus of this section. I shall show that the expected efficiency loss is also on the order of $1/(m \wedge n)$, where $m \wedge n = \min\{m, n\}$ is the minimum of the number of buyers and sellers, provided that the densities are bounded away from zero.

Because dominant strategies are independent of the distribution, a modest amount of generality is available at low overhead cost. Let $\mathscr{F}$ be a family of distributions with support $[0, 1]$ and continuous densities bounded above zero over $[0, 1]$.[7] Suppose distributions F and G are drawn from $\mathscr{F}$ according to some stochastic process. Denote the densities of F and G by f and g, respectively. By assuming the densities of distributions in $\mathscr{F}$ are bounded above zero, I have assumed that:

$$\varphi = \min\{f(x): 0 \leqslant x \leqslant 1\} > 0, \tag{8}$$

$$\gamma = \min\{g(x): 0 \leqslant x \leqslant 1\} > 0. \tag{9}$$

Intuitively, the use of (8) and (9) is to force the order statistics to be of order $1/(m \wedge n)$ apart. If the density approaches zero, gaps may appear in the order statistics that are not closed by additional realizations very rapidly. Moreover, if $b_{(i)} - b_{(i+1)}$ is unusually large, then k is more likely to be i, because it is more likely that $s_{(i)} < b_{(i)}$ and $s_{(i+1)} > b_{(i+1)}$ if $b_{(i)}$ is large and $b_{(i+1)}$ is small. Thus, calculating the efficiency loss is not merely a matter of the expected difference in order statistics.

Valuations and costs are generated as follows. First, distributions F and G are drawn at random from $\mathscr{F}$. Then, valuations $b_1, \ldots, b_m$ are drawn from F and $s_1, \ldots, s_n$ are drawn from G, and these are all drawn

[7] The restriction of support to $[0, 1]$ generalizes to any compact interval. However, it is important to the argument that F and G have the identical interval as support.

independently, conditional on F and G.[8] Define the sample distribution functions:

$$F^m(b) = (m-i)/m \qquad \text{if} \quad b_{(i)} \geqslant b > b_{(i+1)}$$

and

$$G^n(s) = i/n \qquad \text{if} \quad s_{(i)} \leqslant s < s_{(i+1)}.$$

The number of buyers willing to pay the price p is $m(1-F^m(p))$, and similarly the number of sellers willing to sell at price p is $nG^n(p)$. Thus, any market clearing price p satisfies $m(1-F^m(p)) = nG^n(p)$, and the efficient number of traders is $k = nG^n(p)$. The efficiency loss associated with the direct implementation of the dominant strategy auction is

$$\lambda = \begin{cases} 0 & \text{if} \quad p_0 \in [s_{(k)}, b_{(k)}] \\ b_{(k)} - s_{(k)} & \text{if} \quad p_0 \notin [s_{(k)}, b_{(k)}]. \end{cases}$$

The following lemma has a straightforward, brute-force proof located in the Appendix. It does not depend on (8) and (9).

LEMMA 2.

$$E\lambda \leqslant \sum_{i=1}^{m \wedge n} \int_0^1 \binom{m}{i}\binom{n}{i} (1-G(x))^{n-i} F(x)^{m-i} iG(x)^{i-1} g(x)$$
$$\times \int_x^1 (1-F(y))^i \, dy \, dx$$
$$+ \sum_{i=1}^{m \wedge n} \int_0^1 \binom{m}{i}\binom{n}{i} (1-G(x))^{n-i} F(x)^{m-i} i(1-F(x))^{i-1} f(x)$$
$$\times \int_0^x G(y)^i \, dy \, dx.$$

Lemma 2 allows a direct proof of the rate of convergence to efficiency, with a simple expression.

THEOREM 3. $E\lambda \leqslant 1/\varphi(m+1) + 1/\gamma(n+1)$.

[8] This process of generating the valuations is known as conditional independence, and is equivalent to the sequence of valuations being exchangeable, see Kingman [3] and Milgrom and Weber [9]. Observe that this process allows a certain type of correlation between the random variables. I thank a referee for pointing out this free generality.

Proof. Note that

$$\int_x^1 (1-F(y))^i\,dy \leqslant \varphi^{-1}\int_x^1 (1-F(y))^i f(y)\,dy = \varphi^{-1}\frac{(1-F(x))^{i+1}}{i+1} \tag{10}$$

Similarly,

$$\int_0^x G(y)^i\,dy \leqslant \gamma^{-1}\frac{G(x)^{i+1}}{i+1}.$$

This yields, from Lemma 2,

$$\begin{aligned} E\lambda \leqslant{}& \varphi^{-1}\sum_{i=1}^{m\wedge n}\int_0^1 \binom{m}{i}\binom{n}{i}(1-G(x))^{n-i}F(x)^{m-i} \\ &\times\frac{i}{i+1}G(x)^{i-1}g(x)(1-F(x))^{i+1}\,dx \\ &+\gamma^{-1}\sum_{i=1}^{m\wedge n}\int_0^1 \binom{m}{i}\binom{n}{i}(1-G(x))^{n-i}F(x)^{m-i} \\ &\times\frac{i}{i+1}f(x)(1-F(x))^{i-1}G(x)^{i+1}\,dx \\ ={}&\frac{n}{\varphi(m+1)}\sum_{i=1}^{m\wedge n}\int_0^1\binom{m+1}{i+1}(1-F(x))^{i+1}F(x)^{m-i} \\ &\times\binom{n-1}{i-1}G(x)^{i-1}(1-G(x))^{m-i}g(x)\,dx \\ &+\frac{m}{\gamma(n+1)}\sum_{i=1}^{m\wedge n}\int_0^1\binom{n+1}{i+1}G(x)^{i+1}(1-G(x))^{n-i} \\ &\times\binom{m-1}{i-1}(1-F(x))^{i-1}F(x)^{m-i}f(x)\,dx. \end{aligned} \tag{11}$$

Consider the first term in (11); the second is symmetric. For $1 \leqslant i \leqslant n$, define

$$\Gamma_i(x) = \int_0^x \binom{n-1}{i-1} G(y)^{i-1}(1-G(y))^{n-i}g(y)\,dy.$$

For $i \leqslant 0$, define $\Gamma_i(x) = \Gamma_1(x)$, and for $i > n$, define $\Gamma_i(x) = \Gamma_n(x)$. Note that $\Gamma_i(1) = 1/n$, and that $\Gamma_i(x)$ is nonincreasing in i and increasing in x.

Let $0 = x_0 < x_1 < \cdots < x_p = 1$ be a partition of $[0, 1]$. Below, the symbol $\approx$ will signify an approximation that is arbitrarily good, becoming equality as the partition is refined. We have

$$\frac{n}{\varphi(m+1)} \sum_{i=1}^{m \wedge n} \int_0^1 \binom{m+1}{i+1} (1-F(x))^{i+1} F(x)^{m-i}$$
$$\times \binom{n-1}{i-1} G(x)^{i-1} (1-G(x))^{n-i} g(x)\, dx$$
$$\approx \frac{n}{\varphi(m+1)} \sum_{i=1}^{m \wedge n} \sum_{j=1}^{p} \int_{x_{j-1}}^{x_j} \binom{m+1}{i+1} (1-F(x_j))^{i+1} F(x_j)^{m-i}$$
$$\times \binom{n-1}{i-1} G(x)^{i-1} (1-G(x))^{n-i} g(x)\, dx$$
$$= \frac{n}{\varphi(m+1)} \sum_{j=1}^{p} \sum_{i=1}^{m \wedge n} \binom{m+1}{i+1}$$
$$\times (1-F(x_j))^{i+1} F(x_j)^{m-i} [\Gamma_i(x_j) - \Gamma_i(x_{j-1})]$$
$$\leqslant \frac{n}{\varphi(m+1)} \sum_{j=1}^{p} \sum_{i=-1}^{m} \binom{m+1}{i+1}$$
$$\times (1-F(x_j))^{i+1} F(x_j)^{m-i} [\Gamma_i(x_j) - \Gamma_i(x_{j-1})]$$
$$= \frac{n}{\varphi(m+1)} \sum_{i=-1}^{m} \binom{m+1}{i+1} (1-F(x_p))^{i+1} F(x_p)^{m-i} \Gamma_i(x_p)$$
$$+ \frac{n}{\varphi(m+1)} \sum_{j=1}^{p-1} \sum_{i=-1}^{m} \Gamma_i(x_j) \left[\binom{m+1}{i+1} (1-F(x_j))^{i+1} F(x_j)^{m-i} \right.$$
$$\left. - \binom{m+1}{i+1} (1-F(x_{j+1}))^{i+1} F(x_{j+1})^{m-i} \right]$$
$$\leqslant \frac{n}{\varphi(m+1)} \Gamma_{-1}(x_p) = \frac{1}{\varphi(m+1)}.$$

The first inequality uses $\Gamma_i'(x) \geqslant 0$, and the subsequent equality uses $\Gamma_i(x_0) = \Gamma_i(0) = 0$. The second inequality uses the fact that the family of probability distributions *over* i given by $\binom{m+1}{i+1}(1-F(x))^{i+1} F(x)^{m-i}$ is ranked by first-order stochastic dominance in x. Since $\Gamma_i(x)$ is nonincreasing in i, this yields

$$\sum_{i=-1}^{m} \Gamma_i(x_j) \binom{m+1}{i+1} (1-F(x_j))^{i+1} F(x_j)^{m-i}$$
$$\leqslant \sum_{i=-1}^{m} \Gamma_i(x_j) \binom{m+1}{i+1} (1-F(x_{j+1}))^{i+1} F(x_{j+1})^{m-i}.$$

The second line in (11) is symmetric. ∎

Remark 3. Since there are at least $2(m \wedge n)$ potential traders, the expected efficiency loss per potential trader is of order $1/(m \wedge n)^2$.

ORAL DOUBLE AUCTION

The oral implementation of the dominant strategy auction, or oral dominant strategy double auction, is a double auction related to Milgrom and Weber's [8] stylized model of the English Auction. The auction operates in continuous time, starting at $t=0$. At each moment in time t, a state of the system involves bid and asked prices $\beta(t)$ and $\sigma(t)$, and the number of active buyers and sellers $m(t)$ and $n(t)$.[9] Initially, all buyers and sellers are active:

$$m(0)=m,$$

$$n(0)=n.$$

A strategy for either a buyer or a seller is a time to become inactive, as a function of the state of the system $(\beta(t), \sigma(t), n(t), m(t), t)$. It is convenient to think of active buyers and sellers in a room together, remaining active by remaining in the room. Buyers and sellers become inactive by exiting, which is irrevocable. If a buyer (seller) becomes inactive at time t, $m(t)(n(t))$ is decreased by a unit. Bid and asked prices are initially set at the most favorable levels:

$$\beta(0)=\inf\{b: F(b)>0\}$$

and

$$\sigma(0)=\sup\{s: G(s)<1\},$$

for the buyer and seller, respectively. During the play of the game, bid and asked prices are governed by the differential equations:

$$\beta'(t)=\begin{cases}1 & \text{if } m(t)\geqslant n(t)\\ 0 & \text{if } m(t)<n(t)\end{cases}$$

$$\sigma'(t)=\begin{cases}1 & \text{if } n(t)\geqslant m(t)\\ 0 & \text{if } n(t)<m(t).\end{cases}$$

Thus, the bid price rises at a unit rate unless there are fewer buyers than sellers, in which case the bid price is constant. Similarly, asked prices decline at a unit rate unless there are fewer sellers than buyers, in which case it is constant.

[9] I abuse notation by using the same symbol for the total or initial number, and the number as a function of time, for its mnemonic value.

The game ends at the first time T occurring with the following conditions prevailing:

(i) $m(T) = n(T)$ and,

(ii) $\beta(T) \geqslant \sigma(T)$.

At termination, trade occurs at prices $\beta(T)$ and $\sigma(T)$ involving all active buyers and sellers. Buyers pay $\beta(T)$ and sellers are paid $\sigma(T)$.

It is obviously a dominant strategy for a buyer with value b to remain active so long as $b > \beta(t)$, and similarly for a seller to remain active so long as $s < \sigma(t)$.[10]

Once the equilibrium strategies are recognized, the play of the game is easily described. Consider the case $m \leqslant n$. Bid prices rise until $n - m$ buyers are eliminated, with asked prices constant. Then both prices move toward each other until a player is eliminated. If a seller is eliminated, asked prices freeze, and bid prices rise until a buyer is eliminated. Thus, bid prices "walk up" the demand function and asked prices "walk down" the supply function (refer to Fig. 1) from right to left; if one gets ahead, he politely pauses until the other catches up.

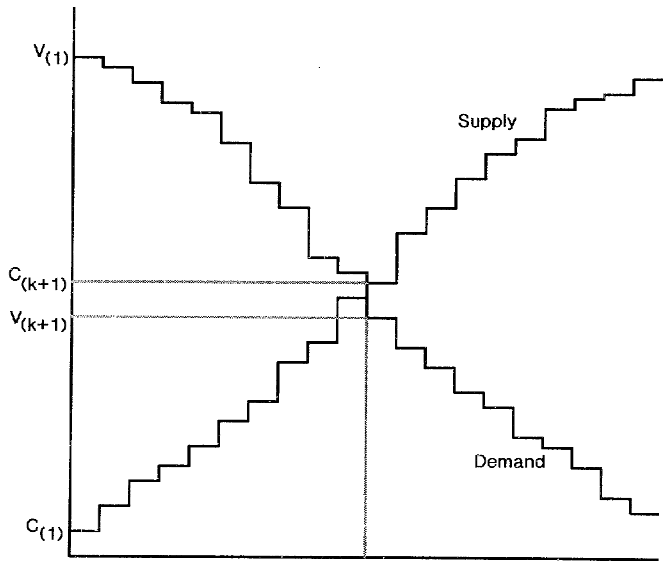

FIGURE 1

[10] This is not the only equilibrium strategy. If $\beta(t) < \sigma(t)$, a player can forecast that the game's termination is at least $\frac{1}{2}(\sigma(t) - \beta(t))$ units of time away, so remaining active is costless. eliminate such behavior by presuming a lexicographic preference for profits over exiting: any positive level of profits is preferred to early exit, while early exit is preferred to later exit. However, all players exiting at $t = 0$ comprises an equilibrium.

This situation prevails until the time t, where

$$\beta(t) = b_{(k+1)} < s_{(k+1)} = \sigma(t),$$

and there are k players left of each type. If

$$s_{(k)} \leqslant \tfrac{1}{2}(b_{(k+1)} + s_{(k+1)}) \leqslant b_{(k)},$$

bid and asked prices will meet in the middle, at p_0, and all k buyers and sellers will be involved in trade. Otherwise, one player will be eliminated before p_0 is reached; the other of player's price will continue to change until one of this type is eliminated as well.[11] This occurs at $\beta(t) = b_{(k)}$ and $\sigma(t) = s_{(k)}$, which ends the game.

This oral auction implements the direct implementation of the dominant strategy auction described in Section 2 and thereby inherits all its convergence properties. The auction is not dissimilar to a one-shot version of the behavior on the floor of the New York Stock Exchange. One important aspect of the NYSE is that, for each stock, there is a market specialist who occasionally intervenes in trade and keeps track of supply and demand at asked and bid prices. Like the mechanism in the present study, the specialist makes money. It is not unreasonable to view the specialist's purchases and sales as buying at $s_{(k)}$ and selling at $b_{(k)}$, to clear the market. Of course, owing to the dynamic aspects of the NYSE, the analogy is stretched, but the oral dominant strategy double auction is as reasonable a model of the NYSE as a sealed-bid double auction. Perhaps more significant for any such analogy, private valuations constitute an unreasonable model of stock valuations.

The oral dominant strategy double auction stylizes the movement of prices in much the same way as Milgrom and Weber [8] stylized the movement of prices for the English (ascending oral) auction, and the assumption of irrevocable exit is analogous as well. The response of prices to excess demand or supply is extreme in the oral dominant strategy double auction, owing to the discrete nature of the good. Alternative price movements, such as rates of change depending continuously on the difference $m(t) - n(t)$, would typically eliminate the dominant strategies.

[11] If two or more such players wish to exit simultaneously, a tie-breaking rule is called for and only one player, chosen at random, eliminated. This rule is only invoked if *preventing* player from exiting will end the game. The tie-breaking rule does not disturb the dominant strategies. Moreover, it ensures that the dominant strategy equilibrium of the oral dominant strategy auction coincides with that of the direct implementation.

CONCLUSION

The double auction model analyzed here possesses some attractive theoretical features. First, the absence of strategic behavior means that the properties of the equilibrium can be established purely on the characteristics of the underlying distribution, without reference to bidding behavior. Second, because at most one transaction is lost, and that is the least valuable, it is immediate that the per trader efficiency loss is of order $1/n$. Finally, both the mechanism and the equilibrium strategies can be defined without reference to the underlying distributions, so that the game and equilibrium allocations are not very sensitive to changes in the distribution of buyer types.

One interesting question concerning the direct implementation of the dominant strategy auction is whether the mechanism's revenue converges to zero. The distribution of the efficient quantity k is difficult to work with; this complexity is inherent in the full information model. It is easily shown that

$$\Pr(k \geqslant i) = \int_0^1 m \binom{m-1}{i-1} (1-F(b))^{i-1} F(b)^{m-i} \times \sum_{j=i}^{m} \binom{n}{j} G(b)^j (1-G(b))^{n-j} \, dF(b)$$

which appears more or less useless. Worse still, the value $b_{(k)} - s_{(k)}$ is correlated with k. In particular, if $b_{(i)} - s_{(i)}$ is unusually large, it is more likely that $b_{(i+1)} - s_{(i+1)} > 0$ and thus that $k \geqslant i+1$. This correlation makes an analysis of $(k-1)(b_{(k)} - s_{(k)})$ very difficult. In addition, $E(k-1)$ is of order n, while $E(b_{(k)} - s_{(k)})$ is of order $1/n$, so that a simple rate argument is not possible.

For the case when both F and G are the same uniform distribution, I have simulated the distributions of outcomes with various values of $n = m$, the results of which are summarized in Table I. It would appear that the probability that $p_0 \in [s_{(k)}, b_{(k)}]$ is declining in $n = m$, but whether it converges to zero is uncertain. It also appears that the gain to the mechanism, $(k-1)(b_{(k)} - s_{(k)})$, is increasing in $n = m$. When divided by $\log(\log n)$, this term appears to decrease. Whether $(k-1)(b_{(k)} - s_{(k)})$ converges or not remains an open question. It should also be noted that the direct implementation of the dominant strategy auction appears to perform significantly worse than the buyer's bid double auction, for small numbers of traders, given the simulations reported by Satterthwaite and Williams. However, the efficiency of the direct implementation of the dominant strategy auction exceeds the efficiency of the double dutch auction found experimentally by McCabe, Rassenti, and Smith [7].

There are at least two extensions of the double auction environment that

TABLE I

The Results of Simulations of the Mechanism, Given Identical Uniform Distributions and Equal Numbers of Buyers and Sellers

$n = m$	Full surplus (percentage)	Mechanism's earnings (%)	Percentage of surplus lost	Buyer's bid percentage lost
2	64.66	.0122 (3.06)	17.741	7.415
3	59.57	.0450 (6.48)	11.079	3.203
4	56.97	.0737 (8.29)	7.593	1.748
5	55.32	.0971 (8.56)	5.417	0.263
10	52.48	.1576 (6.61)	1.625	na
15	51.86	.1843 (5.08)	0.776	na
25	51.29	.2078 (3.39)	0.295	na
50	50.75	.2273 (1.84)	0.0764	na
100	50.68	.2365 (0.95)	0.0194	na
500	50.10	.2476 (0.20)	0.0008	na
1000	50.26	.2481 (0.10)	0.0002	na

Note. The first column is the number of buyers and sellers. The second column provides the percentage of times the mechanism achieved the full information efficient solution. The third column gives the average earnings by the mechanism, both absolutely and as a percentage of the total surplus. The fourth column is the average value of nonexecuted trades divided by the average full information gains from trade, in percent. For each value of n, there were 50,000 double auctions simulated. Finally, the fifth column gives the percentage losses for the buyer's bid double auction, as reported by Satterthwaite and Williams [10], under the same distributional assumptions.

would be quite valuable. First, trade executed by double auctions typicall involves multiple units, and a proper analysis with multiple units per trade would be interesting. The direct implementation of the dominant strateg auction loses its dominant strategy property when multiple units are introduced, because a buyer would be tempted to lower his report on one unit hoping to obtain a lower price for other units he buys. McAfee [5] ha shown that continuous quantities may permit implementation of efficien allocations, and thus multiple units may improve the situation.

The second issue is private values and correlation in values. This pape dealt with correlated values to a very limited extent, and did not conside generalizations outside of private values. Yet, as Milgrom and Weber [8 persuasively argue, private values is an implausible assumption, failing i for example, buyers have private information about the durability or resal value of a durable good. An analysis offering the generality for the doubl auction that Milgrom and Weber brought to one sided auctions appear intractable. However, McAfee and Reny [6] offer a mechanism desig approach to such problems in the presence of risk neutrality and correlate signals.

APPENDIX

Proof of Lemma 2. First, note that if $\lambda > 0$, that is, if $p_0 \notin [s_{(k)}, b_{(k)}]$, then either (i) $b_{(k+1)} < s_{(k)}$ or (ii) $s_{(k+1)} > b_{(k)}$, or both. Let 1 be the characteristic function, so that 1_A is 1 on the set A, and 0 otherwise. For $i < m$,

$$\begin{aligned}
&E\{(b_{(i)} - s_{(i)})\, 1_{\{b_{(i+1)} < s_{(i)} \leqslant b_{(i)}\}}\} \\
&= E\left\{(b_{(i)} - s_{(i)}) \left(\frac{F(s_{(i)})}{F(b_{(i)})}\right)^{m-i} 1_{\{s_{(i)} \leqslant b_{(i)}\}}\right\} \\
&= \int_0^1 i\binom{n}{i} G(s)^{i-1} (1-G(s))^{n-i} g(s) \\
&\quad \times \int_s^1 (b-s) \left(\frac{F(s)}{F(b)}\right)^{m-i} i\binom{m}{i} (1-F(b))^{i-1} F(b)^{m-i} f(b)\, db\, ds \\
&= \int_0^1 i\binom{n}{i} G(s)^{i-1} (1-G(s))^{n-i} g(s) F(s)^{m-i} \\
&\quad \times \int_s^1 (b-s) i\binom{m}{i} (1-F(b))^{i-1} f(b)\, db\, ds \\
&= \int_0^1 i\binom{n}{i} G(s)^{i-1} (1-G(s))^{n-i} g(s) F(s)^{m-i} \int_s^1 \binom{m}{i} (1-F(b))^{i}\, db\, ds.
\end{aligned}$$

For $i = m \leqslant n$, this evaluates to $E\{b_{(i)} - s_{(i)}\}$. Similarly,

$$\begin{aligned}
E\{(b_{(i)} - s_{(i)})\, 1_{\{s_{(i+1)} > b_{(i)} \geqslant s_{(i)}\}}\} &= \int_0^1 i\binom{m}{i} (1-F(b))^{i-1} F(b)^{m-i} f(b) \\
&\quad \times \binom{n}{i} (1-G(b))^{n-i} \int_0^b G(s)^i\, ds\, db.
\end{aligned}$$

Finally,

$$\begin{aligned}
&E\{(b_{(k)} - s_{(k)})\, 1_{\{p_0 \notin [s_{(k)}, b_{(k)}]\}}\} \\
&\quad \leqslant E\{(b_{(k)} - s_{(k)})\, 1_{\{b_{(k+1)} < s_{(k)}\} \cup \{s_{(k+1)} \geqslant b_{(k)}\}}\} \\
&\quad = \sum_{i=1}^{m \wedge n} E\{(b_{(i)} - s_{(i)})\, 1_{\{b_{(i+1)} < s_{(i)} \leqslant b_{(i)}\} \cup \{s_{(i+1)} > b_{(i)} \geqslant s_{(i)}\}}\} \\
&\quad \leqslant \sum_{i=1}^{m \wedge n} E\{(b_{(i)} - s_{(i)})(1_{\{b_{(i+1)} < s_{(i)} \leqslant b_{(i)}\}} + 1_{\{s_{(i+1)} > b_{(i)} \geqslant s_{(i)}\}})\}.
\end{aligned}$$

This completes the proof. ∎

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