

## COARSE MATCHING

BY R. PRESTON MCAFEE<sup>1</sup>

Keywords: Priority pricing, matching, hazard rate, blackout, rolling blackout.

California recently used “rolling blackouts” as a means of rationing electricity. In this arrangement, a few high priority customers such as hospitals are excluded from rationing, but otherwise customers experience short-lived blackouts without regard to the social or private importance of their electricity use. Are rolling blackouts that affect virtually every user a good strategy? Is there a simple way to improve on the random blackout?

Oren, Smith and Wilson (1985) and Wilson (1989) consider means of mitigating the social consequences of rationing by matching the high value users to the power provision. Essentially, the goal is to use *ex ante* pricing, or priority pricing, to insure that high value users have a higher probability of receiving the good. In these papers, which differ by whether the pricing is monopolistic or competitive, demand is characterized by a distribution  $F$  of values, and a distribution  $H$  of total units available. If a customer with value  $x$  obtains a probability of service  $y$ , the total surplus generated is  $xy$ .

The first best solution to the allocation problem involves spot markets, which permit the rationing of the available supply to the highest value customers. Spot markets, however, are not always feasible – it may be difficult or impossible to contact customers after the realization of supply, especially when supply disruptions are abrupt. It is possible to implement the spot market solution using priority pricing. Priority pricing orders customers from highest to lowest based on customer payments for service reliability, and provides the good in that order – e.g. if five units are available, the top five in the priority queue obtain the good. With a continuous distribution of availability, full priority pricing requires a continuum of positions in the queue and associated priority prices.

The use of a continuum of priorities is not feasible in many circumstances – using many priorities makes the scheme unwieldy to administer and opaque to consumers. Moreover, if the

priority prices are determined by bidding, as is natural, the auction process will be complex and expensive to operate when there are many service classes. Instead, it is desirable to use a limited number of priority classes, perhaps just two. The two-class scheme involves two priorities, high and low. When capacity is insufficient to supply all the high priority customers, capacity is randomly allocated among high priority customers. When capacity is insufficient to serve all customers, but more than sufficient to serve the high priority customers, all high priority customers are served, with the balance randomly allocated among low priority customers. High priority customers pay a common priority price for their privileged position. How well does such a scheme perform? The mathematical fact proved in this paper is that if the distributions satisfy standard hazard rate hypotheses, the social value of using two priorities is at least half of the gains of using infinitely many priority classes. Thus, the social cost of using simplified priority pricing schemes may be modest, and outweighed by the benefits of simplicity and transparency.

Smith's 1995 marriage model<sup>2</sup> provides a convenient illustration of the theory. In this model, the value of a match is the product of the types of the agents, who can be taken to be men and women. Suppose that the types are random variables  $X$  and  $Y$ , and that these are uniformly distributed on  $[0,1]$ . Consider various methods for matching men and women. Using a purely random assignment of men to women, the average match has a value of  $u_0 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ : the average value of  $X$  is matched to the average value of  $Y$ . Perfect matching maximizes the average value of a match by associating identical values of  $X$  and  $Y$ , for an average value of

$$u_\infty = \int_0^1 x^2 dx = \frac{1}{3}.$$

If two equal-sized classes are used, randomly chosen elements in the bottom half (mean  $\frac{1}{4}$ ) are matched with others in the bottom half, and randomly chosen elements in the top half (mean  $\frac{3}{4}$ ) are matched with others in the top half, to produce an overall value<sup>3</sup> of

$$u_1 = \frac{1}{2} \left( \left( \frac{1}{4} \right)^2 + \left( \frac{3}{4} \right)^2 \right) = \frac{5}{16}.$$

Using two classes obtains 75% of the value of using infinitely many priority classes, that is,

$$\frac{u_1 - u_0}{u_\infty - u_0} = \frac{3}{4}.$$

The next section generalizes the theory beyond uniform distributions, and shows how priority pricing can be interpreted as a matching model.

### 1. Mathematical Formulation

Suppose  $X$  and  $Y$  are two random variables with distributions  $F$  and  $G$ , respectively. These distributions are assumed to have continuous densities  $f$  and  $g$ , and the densities have interval support. Let  $\mu_X$  and  $\mu_Y$  be the means of  $X$  and  $Y$ . I assume both  $F$  and  $G$  have finite variances, to insure convergence of integrals.

The random variables are perfectly matched if low values of one are associated with low values of the other. Define the function  $\psi$  by  $G(\psi(x))=F(x)$ .

Perfect matching involves matching the value  $x$  with the value  $\psi(x)$  drawn from  $G$ . This matches quantiles of the distributions, so that high values of one variable are associated as best possible with high values of the other distribution. The expected value from perfect matching is:

$$u_\infty = \int_{-\infty}^{\infty} x\psi(x)f(x)dx.$$

Random matching associates each draw from one distribution with a random draw from the other, and thus has the overall value  $u_0 = \mu_X \mu_Y$ .

Matching with two priority classes and a cutoff point of  $x^*$  means associating  $X$  values below  $x^*$  with  $Y$  values below  $\psi(x^*)$ . The function  $\psi$  was chosen so that these have the same frequency, that is,  $F(x^*) = G(\psi(x^*))$ . In this case, the social value of matching is:

$$u_1 = F(x^*) \int_{-\infty}^{x^*} x \frac{f(x)}{F(x^*)} dx \int_{-\infty}^{\psi(x^*)} y \frac{g(y)}{G(\psi(x^*))} dy + (1 - F(x^*)) \int_{x^*}^{\infty} x \frac{f(x)}{1 - F(x^*)} dx \int_{\psi(x^*)}^{\infty} y \frac{g(y)}{1 - G(\psi(x^*))} dy$$

The value  $u_1$  arises because low  $X$  values, below  $x^*$ , are associated with low  $Y$  values. Thus, the average match value is the probability of low values  $F(x^*)$  times the expected value of the low value match, which is the product of the expected values conditional on being low, plus the similar term conditional on both being high. This association is perfect in the sense that no low  $X$  values are ever associated with high  $Y$  values, but imperfect in the sense of using only two priority classes.

*Priority Pricing:* Consider the electricity example of Wilson (1989). Some work is required to map this example into the framework. There is a distribution of values for the good or service denoted  $F$ , so that  $1 - F(v)$  of the population have value for the good exceeding  $v$ .<sup>4</sup> There is a distribution of available power  $H$ . The natural distributions are in conflicting units – one in values, one in power – and to streamline the analysis, it is helpful to convert the quantity units of electricity supply into population service quantiles, that is, into a distribution of service probabilities.<sup>5</sup> Under priority pricing, a customer with value  $v$  is served if there is enough power to serve all the customers with value at least  $v$ , which requires a quantity  $q \geq N(1 - F(v))$ , where  $N$  is the total number of customers. The probability the customer of value  $v$  is served is the

probability that  $q \geq N(1-F(v))$ , which is  $1-H(N(1-F(v)))$ .<sup>6</sup> In the electricity example, this is the function  $\psi$ , that is,  $\psi(v)= 1-H(N(1-F(v)))$ .

The distribution of service probabilities under priority pricing, then, is the function  $G$  solving  $G(\psi(v))=F(v)$ , or  $G(1-H(N(1-y)))=y$ . Note that  $G$  does not depend on  $F$ :  $G$  is the feasible distribution of service priorities and its units are quantiles, not values. If each customer is served with a probability independent of the customer's value, the probability that a customer is served is  $q/N$  when there is a supply of  $q$ , so that the overall probability of being served is:

$$\begin{aligned} \int_0^N \frac{q}{N} h(q) dq &= \int_0^1 (1-y) h(N(1-y)) N dy = 1 - \int_0^1 H(N(1-y)) dy = \int_0^1 1 - H(N(1-y)) dy \\ &= \int_0^\infty 1 - H(N(1-F(v))) f(v) dv = \int_0^\infty \psi(v) f(v) dv = \int_0^\infty y g(y) dy. \end{aligned}$$

That is, when the priority assignments are not used, the probability of service is just the average of the probabilities of service, weighted by the frequency of types. Such a proposition is true with priority classes as well. For example, suppose that customers with values exceeding  $v^*$  are given priority over those with values below  $v^*$ . Priority means no customer with value below  $v^*$  is served when a customer with value above is unserved. In this case, the probability of being served for a customer with value exceeding  $v^*$  is

$$\begin{aligned} &\int_0^{N(1-F(v^*))} \frac{q}{N(1-F(v^*))} h(q) dq + \int_{N(1-F(v^*))}^N 1 \times h(q) dq \\ &= H(N(1-F(v^*))) - \int_0^{N(1-F(v^*))} \frac{H(q)}{N(1-F(v^*))} dq + 1 - H(N(1-F(v^*))) \\ &= \int_0^{N(1-F(v^*))} \frac{1-H(q)}{N(1-F(v^*))} dq = \int_{v^*}^\infty 1 - H(N(1-F(v))) \frac{Nf(v)}{N(1-F(v^*))} dv \end{aligned}$$

$$= \int_{v^*}^{\infty} \psi(v) \frac{f(v)}{1 - F(v^*)} dv = \int_{\psi(v^*)}^{\infty} y \frac{g(y)}{1 - G(\psi(v^*))} dy$$

Thus, the average probability of service in the high group is just the average of the probabilities of service of the high group under priority pricing.

The above analysis demonstrates a connection between the priority pricing problem and the matching problem. From the primitive  $H$ , the distribution of power supply, there is a distribution of service probabilities  $G$ , which can be matched to customers in exactly the same way that the marriage model matches the types of men and women. The distribution  $G$  of service probabilities satisfies  $G(1 - H(N(1 - y))) = y$ .

## 2. Theorem and Proof

The theorem will require that two commonly used hazard rate conditions apply to both distributions. These are:

- (1)  $\frac{F(x)}{f(x)}$  and  $\frac{G(y)}{g(y)}$  are both increasing, and
- (2)  $\frac{1 - F(x)}{f(x)}$  and  $\frac{1 - G(y)}{g(y)}$  are decreasing.

These conditions are frequently encountered in adverse selection problems, and they insure that the density does not locally vary too much. When  $F$  corresponds to the distribution of failure times, (2) is equivalent to the requirement that the likelihood of failure conditioned on no failure having arisen is an increasing function. For example, the probability of a light bulb failing in the next minute, given that it hasn't yet failed, grows as the light bulb ages.<sup>7</sup>

THEOREM: *If (1) and (2) hold, and  $x^* = \mu_X$ , or  $\psi(x^*) = \mu_Y$ , then  $\frac{u_1 - u_0}{u_\infty - u_0} \geq \frac{1}{2}$ .*

PROOF: The structure of the proof is simple, although the proof is quite long. The basic structure is to rewrite  $u_\infty$ , apply the Cauchy-Schwarz inequality, then recombine so that the theorem is obvious. The proof starts with a change of variables that dramatically simplifies the analysis, utilizing the inverses of the distribution functions. Note that  $zF^{-1}'(z)$  is increasing for  $z$  in  $[0,1]$  if and only if  $F(x)F^{-1}'(F(x)) = \frac{F(x)}{f(x)}$  is increasing for  $x$  in the support of  $f$ , and

similarly  $(1-z)F^{-1}'(z)$  is decreasing if and only if  $(1-F(x))F^{-1}'(F(x)) = \frac{1-F(x)}{f(x)}$  is

decreasing. Thus, (1) and (2) become:

$$(3) \quad zF^{-1}'(z) \text{ and } zG^{-1}'(z) \text{ are increasing for } z \text{ in } [0,1], \text{ and}$$

$$(4) \quad (1-z)F^{-1}'(z) \text{ and } (1-z)G^{-1}'(z) \text{ are decreasing for } z \text{ in } [0,1].$$

Using the change of variables  $z=F(x)$  or  $z=G(y)$ , note that:

$$u_\infty = \int_{-\infty}^{\infty} x\psi(x)f(x)dx = \int_{-\infty}^{\infty} xG^{-1}(F(x))f(x)dx = \int_0^1 F^{-1}(z)G^{-1}(z)dz.$$

$$u_0 = \int_{-\infty}^{\infty} xf(x)dx \int_{-\infty}^{\infty} yg(y)dy = \int_0^1 F^{-1}(z)dz \int_0^1 G^{-1}(z)dz = \mu_X \mu_Y$$

and

$$\begin{aligned} u_1 &= F(x^*) \int_{-\infty}^{x^*} x \frac{f(x)}{F(x^*)} dx \int_{-\infty}^{\psi(x^*)} y \frac{g(y)}{G(\psi(x^*))} dy + (1-F(x^*)) \int_{x^*}^{\infty} x \frac{f(x)}{1-F(x^*)} dx \int_{\psi(x^*)}^{\infty} y \frac{g(y)}{1-G(\psi(x^*))} dy \\ &= c \int_0^c F^{-1}(z) \frac{dz}{c} \int_0^c G^{-1}(z) \frac{dz}{c} + (1-c) \int_c^1 F^{-1}(z) \frac{dz}{1-c} \int_c^1 G^{-1}(z) \frac{dz}{1-c} \end{aligned}$$



$$= \frac{1}{c} \int_0^c F^{-1}(z) dz \int_0^c G^{-1}(z) dz + \frac{1}{1-c} \int_c^1 F^{-1}(z) dz \int_c^1 G^{-1}(z) dz,$$

where  $c=F(x^*)=G(\psi(x^*))$ .

Moreover, it is immediate that

$$u_1 - u_0 = \frac{1}{c} \int_0^c (F^{-1}(z) - \mu_X) dz \int_0^c (G^{-1}(z) - \mu_Y) dz + \frac{1}{1-c} \int_c^1 (F^{-1}(z) - \mu_X) dz \int_c^1 (G^{-1}(z) - \mu_Y) dz.$$

The key step in the analysis below involves the inequality

$$(5) \quad \int_0^c z G^{-1'}(z) (F^{-1}(z) - \mu_X) \frac{dz}{c} \geq \int_0^c z G^{-1'}(z) \frac{dz}{c} \times \int_0^c (F^{-1}(z) - \mu_X) \frac{dz}{c},$$

which follows from the fact that  $zG^{-1'}(z)$  and  $F^{-1}(z) - \mu_X$  are both increasing functions of  $z$ , and hence positively correlated, and thus the expected value of the product exceeds the product of the expected values.<sup>8</sup> This inequality is used in three other guises below, but the logic is the same. A second fact, used repeatedly, is

$$\mu_X = \int_0^1 F^{-1}(z) dz \quad \text{and} \quad \mu_Y = \int_0^1 G^{-1}(z) dz.$$

Rewrite  $u_\infty$  to obtain:

$$\begin{aligned} u_\infty - u_0 &= \int_0^1 F^{-1}(z) G^{-1}(z) dz - \int_0^1 F^{-1}(z) dz \int_0^1 G^{-1}(z) dz = \int_0^1 (F^{-1}(z) - \mu_X)(G^{-1}(z) - \mu_Y) dz \\ &= \int_0^c (F^{-1}(z) - \mu_X)(G^{-1}(z) - \mu_Y) dz + \int_c^1 (F^{-1}(z) - \mu_X)(G^{-1}(z) - \mu_Y) dz \end{aligned}$$

(Integrate by parts to obtain:)

$$\begin{aligned}
&= c(F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) - \int_0^c zF^{-1}'(z)(G^{-1}(z) - \mu_Y)dz - \int_0^c zG^{-1}'(z)(F^{-1}(z) - \mu_X)dz \\
&\quad + (1-c)(F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) + \int_c^1 (1-z)F^{-1}'(z)(G^{-1}(z) - \mu_Y)dz \\
&\quad + \int_c^1 (1-z)G^{-1}'(z)(F^{-1}(z) - \mu_X)dz
\end{aligned}$$

(Collect terms and multiply and divide by  $c$  or  $1-c$ )

$$\begin{aligned}
&= (F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) - c \int_0^c zF^{-1}'(z)(G^{-1}(z) - \mu_Y) \frac{dz}{c} - c \int_0^c zG^{-1}'(z)(F^{-1}(z) - \mu_X) \frac{dz}{c} \\
&\quad + (1-c) \int_c^1 (1-z)F^{-1}'(z)(G^{-1}(z) - \mu_Y) \frac{dz}{1-c} + (1-c) \int_c^1 (1-z)G^{-1}'(z)(F^{-1}(z) - \mu_X) \frac{dz}{1-c}
\end{aligned}$$

(Employ four versions of (5))

$$\begin{aligned}
&\leq (F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) - c \int_0^c zF^{-1}'(z) \frac{dz}{c} \times \int_0^c (G^{-1}(z) - \mu_Y) \frac{dz}{c} \\
&\quad - c \int_0^c zG^{-1}'(z) \frac{dz}{c} \times \int_0^c (F^{-1}(z) - \mu_X) \frac{dz}{c} \\
&\quad + (1-c) \int_c^1 (1-z)F^{-1}'(z) \frac{dz}{1-c} \times \int_c^1 (G^{-1}(z) - \mu_Y) \frac{dz}{1-c} \\
&\quad + (1-c) \int_c^1 (1-z)G^{-1}'(z) \frac{dz}{1-c} \times \int_c^1 (F^{-1}(z) - \mu_X) \frac{dz}{1-c}
\end{aligned}$$

(Integrate by parts and collect terms)

$$\begin{aligned}
&= (F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) - (F^{-1}(c) - \mu_X) \int_0^c (G^{-1}(z) - \mu_Y) dz \\
&\quad + \frac{1}{c} \int_0^c (F^{-1}(z) - \mu_X) dz \times \int_0^c (G^{-1}(z) - \mu_Y) dz \\
&\quad - (G^{-1}(c) - \mu_Y) \int_0^c (F^{-1}(z) - \mu_X) dz + \frac{1}{c} \int_0^c (G^{-1}(z) - \mu_Y) dz \times \int_0^c (F^{-1}(z) - \mu_X) dz \\
&\quad + (F^{-1}(c) - \mu_X) \int_c^1 (G^{-1}(z) - \mu_Y) dz + \frac{1}{1-c} \int_c^1 (F^{-1}(z) - \mu_X) dz \times \int_c^1 (G^{-1}(z) - \mu_Y) dz \\
&\quad + (G^{-1}(c) - \mu_Y) \int_c^1 (F^{-1}(z) - \mu_X) dz + \frac{1}{1-c} \int_c^1 (G^{-1}(z) - \mu_Y) dz \int_c^1 (F^{-1}(z) - \mu_X) dz
\end{aligned}$$

(Collect terms)

$$\begin{aligned}
&= (F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) + \frac{2}{c} \int_0^c (G^{-1}(z) - \mu_Y) dz \times \int_0^c (F^{-1}(z) - \mu_X) dz \\
&\quad + \frac{2}{1-c} \int_c^1 (F^{-1}(z) - \mu_X) dz \times \int_c^1 (G^{-1}(z) - \mu_Y) dz \\
&= (F^{-1}(c) - \mu_X)(G^{-1}(c) - \mu_Y) + 2(u_1 - u_0)
\end{aligned}$$

Thus if  $c=F(\mu_X)$ , or if  $c=G(\mu_Y)$ ,  $u_\infty - u_0 \leq 2(u_1 - u_0)$ .

Q.E.D.

*Remark 1:* A corollary of the proof is that the worst distribution for two classes is two-tailed exponential. This is a direct consequence of the hazard rate requirement that binds everywhere, yielding equality in the theorem.

*Remark 2:* When  $F$  and  $G$  are both uniform, the use of two classes yields 75% of the potential from matching. When both are one-tailed exponential, the use of two classes produces

74.62%. When both are normally distributed, two classes result in a payoff of  $2/\pi$  or about 63% of the possible gains.

*Remark 3:* When  $F=G$  and  $f$  is symmetric around zero,  $u_0=0$  and  $F(-x)=1-F(x)$ . Moreover,

$$\int_{-\infty}^{x^*} xf(x)dx = - \int_{x^*}^{\infty} xf(x)dx. \text{ When the critical value } x^* \geq 0 \text{ is used, and } c=F(x^*):$$

$$(6) \quad u_1 = \frac{\left( \int_{-\infty}^{x^*} xf(x)dx \right)^2}{F(x^*)} + \frac{\left( \int_{x^*}^{\infty} xf(x)dx \right)^2}{1-F(x^*)} = \frac{\left( \int_{x^*}^{\infty} xf(x)dx \right)^2}{F(x^*)(1-F(x^*))} = \frac{\left( \int_c^1 F^{-1}(y)dy \right)^2}{c(1-c)}.$$

$$\text{Moreover, } u_{\infty} = 2 \int_{\frac{1}{2}}^1 \left( F^{-1}(y) \right)^2 dy.$$

The formula (6) is valuable because it permits us to handle an asymmetric critical point. Note that  $u_1$  is symmetric around 0, so we can concentrate on  $x^* \geq 0$ , or  $c \geq \frac{1}{2}$ . Even though the mean type has been set to zero, the formula (6) incorporates a classification with groups  $(-\infty, x^*)$  and  $[x^*, \infty)$ , where  $x^*$  may not be 0. In fact, it is often the case that the value of  $x^*$  maximizing  $u_1$  is not the mean, as the next remark illustrates.

*Remark 4:* The hazard rate hypotheses are critical to the theorem. Consider the family of distribution functions  $F$  given by  $F^{-1}(y) = (1-y)^{-b} - 2^b$ , for  $0 < b < \frac{1}{2}$ , and  $y > \frac{1}{2}$ . This corresponds to  $F(x) = 1 - (2^b + x)^{-1/b}$ , for  $x \geq 0$ . Values of  $F$  for  $x < 0$  are obtained by symmetry, positing  $F(-x) = 1 - F(x)$ , and let  $G = F$ , so that Remark 3 can be used. This family of distributions strongly violates the hazard rate condition, as can be seen by inspection of (4). It is a routine computation to show that

$$u_{\infty} = 2 \int_{\frac{1}{2}}^1 \left( F^{-1}(y) \right)^2 dy = \frac{2^{2b+1} b^2}{(1-2b)(1-b)}, \text{ and,}$$

$$u_1 = \frac{\left( \int_c^1 F^{-1}(y) dy \right)^2}{c(1-c)} = \frac{1}{c(1-c)} \left( \frac{(1-c)^{1-b}}{1-b} - (1-c)2^b \right)^2 = \frac{1}{c} \left( \frac{(1-c)^{\frac{1}{2}-b}}{1-b} - \sqrt{1-c}2^b \right)^2.$$

As  $b \rightarrow \frac{1}{2}$ ,  $u_\infty$  diverges, recalling  $c \geq \frac{1}{2}$ ,

$$u_1 \rightarrow \frac{(2 - \sqrt{2}\sqrt{1-c})^2}{c} \leq 4.$$

Thus, as  $b$  gets close to  $\frac{1}{2}$ , the best critical value converges to infinity ( $F(x^*)=c \rightarrow 1$ ) and  $u_1/u_\infty \rightarrow 0$ . Intuitively, the value of separating the very highest types from the middle types has increasing value with this distribution as  $b$  grows toward  $\frac{1}{2}$ , and a two-class matching scheme can only separate the highest from the middle, but then lumps the lowest types (which are symmetric) with the middle.

*Remark 5:* If  $F(\mu_X) < G(\mu_Y)$ , then any critical point  $x^* \in [\mu_X, F^{-1}(G(\mu_Y))]$  will satisfy the theorem.

### 3. Conclusion

*There are two types of people: those that divide people into two types, and those that don't.*

– Source unknown

This paper examined the value of using two priority classes in matching and rationing problems, and found that a majority of the gains created by complex schemes are available using very coarse schemes. The result is applicable in a variety of contexts, including the Laffont & Tirole's 1986 principal agent model.<sup>9</sup>

The proof does not generalize naturally to more than two classes, although this may be due to an absence of obvious candidates for hazard rate assumptions. Wilson (1989) shows that the losses are of order  $1/n^2$  for  $n$  priority classes.

The mean of either distribution can usefully serve as a cutoff point. This is an advantage of the theory, because in some circumstances estimates of the mean will be available. In particular, offering the two classes “above average” and “below average” does at least half as well as a continuum of service classes.

It is rare to see more than a small number of classes of service, product types or levels of quantity discounts. This could be a consequence of the costs of offering multiple classes. However, the present analysis suggests that the losses associated with a small number of priority classes are modest at best.

*Department of Economics, University of Texas, Austin, TX 78712;*  
*mcafee@eco.utexas.edu; <http://www.eco.utexas.edu/~mcafee/index.html>*

## References

- LAFFONT, J.-J. AND J. TIROLE (1986): "Using Cost Information to Regulate Firms," *Journal of Political Economy*, 94, 614-41.
- LU, X., AND R. P. MCAFEE (1996): "Matching and Expectations in a Market with Heterogeneous Agents," *Advances in Applied Micro-Economics, Volume 6*, ed: Michael Baye, Greenwich, CT: JAI Press.
- MILGROM, P. AND R. WEBER (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089-1122.
- OREN, S., S. SMITH AND R. WILSON (1985): "Capacity Pricing," *Econometrica*, 53, May 1985, 545-566.
- ROGERSON, W. (2001), "Simple Menus of Contracts in Cost-Based Procurement and Regulation," Unpublished Manuscript, Northwestern University.
- SMITH, L. (1995): "Cross-Sectional Dynamics in a Two-Sided Matching Model," Unpublished Manuscript, University of Michigan.
- WILSON, R. (1989): "Efficient and Competitive Rationing," *Econometrica*, 57, 1-40.

APPENDIX A: COVARIANCE OF NON-DECREASING FUNCTIONS NON-NEGATIVE

Let  $Y, Z$  be random variables formed by nondecreasing functions  $y, z$  of a real-valued random variable  $X$ . Let  $F$  be the distribution of  $X$ . Suppose that the variances of  $Y$  and  $Z$  are finite. Then  $\text{COV}(Y, Z) \geq 0$ .

Proof: Let  $\mu_Y, \mu_Z$  be the means of the two random variables. Define  $u$  by

$$u(x) = \int_{\mu_Z}^x z(x) - \mu_Z dF(x).$$

Note that  $u$  is nonincreasing for  $x < \mu_Z$ , and nondecreasing otherwise. Thus,  $u$  is minimized at  $x = \mu_Z$ . Moreover,

$$0 = \int_{-\infty}^{\infty} z(x) - \mu_Z dF(x) = \int_{-\infty}^{\mu_Z} z(x) - \mu_Z dF(x) + \int_{\mu_Z}^{\infty} z(x) - \mu_Z dF(x) = -u(-\infty) + u(\infty).$$

(The hypothesis that the variance of  $Z$  is finite insures that the terms  $u(\infty)$  and  $u(-\infty)$  are finite.)

For the moment, suppose  $y$  is continuously differentiable.



$$\begin{aligned}
COV(Y, Z) &= \int_{-\infty}^{\infty} y(x)(z(x) - \mu_Z) dF(x) = \int_{-\infty}^{\infty} y(x) du(x) \\
&= u(x)y(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)y'(x) dx = u(\infty)y(\infty) - u(-\infty)y(-\infty) - \int_{-\infty}^{\infty} u(x)y'(x) dx \\
&= u(\infty)(y(\infty) - y(-\infty)) - \int_{-\infty}^{\infty} u(x)y'(x) dx = \int_{-\infty}^{\infty} (u(\infty) - u(x))y'(x) dx \geq 0.
\end{aligned}$$

When  $y$  is not differentiable,  $y$  can be approximated by a differentiable function to obtain the same result.

## APPENDIX B: LAFFONT-TIROLE APPLICATION

Let the output share to an agent be  $\alpha(x)$ , where  $x$  is the agent's type, output is the product of type  $x$  and effort  $e$ , and the cost of effort is  $\frac{1}{2} e^2$ . The agent with type  $x$  and share  $\alpha$  maximizes utility  $v = \alpha x e + \beta - \frac{1}{2} e^2$  by choosing  $e = \alpha x$ , which gives a utility  $v = \frac{1}{2} (\alpha x)^2 + \beta$ , where  $\beta$  is a lump-sum payment. Generally  $\beta$  will depend on  $\alpha$ , and the agent will choose  $\alpha$  to maximize  $v$ . If the principal cannot observe  $x$ , the envelope theorem applied to the choice of  $\alpha$  yields  $\frac{dv}{d\alpha} = \alpha^2 x$ .

The principal obtains

$$\begin{aligned}
 \pi &= \int ((1 - \alpha)x e - \beta) f(x) dx = \int (x e - \frac{1}{2} e^2 - u) f(x) dx = \int (\alpha x^2 - \frac{1}{2} (\alpha x)^2 - u) f(x) dx \\
 &= \int \left( \alpha x^2 - \frac{1}{2} (\alpha x)^2 - \alpha^2 x \frac{1 - F(x)}{f(x)} \right) f(x) dx \\
 &= \int \left( \alpha - \frac{1}{2} \alpha^2 - \alpha^2 \frac{1 - F(x)}{x f(x)} \right) x^2 f(x) dx \\
 &= \left( \int y^2 f(y) dy \right) \int \left( \alpha - \frac{1}{2} \alpha^2 - \alpha^2 \frac{1 - F(x)}{x f(x)} \right) \frac{x^2 f(x)}{\int y^2 f(y) dy} dx \\
 &= \left( \int y^2 f(y) dy \right) \left[ E[\alpha] - E \left[ \alpha^2 \left( \frac{1}{2} - \frac{1 - F(x)}{x f(x)} \right) \right] \right].
 \end{aligned}$$

The expectation  $E$  is defined with respect to the density  $g(x) = \frac{x^2 f(x)}{\int y^2 f(y) dy}$ . The theorem can,

in principle, be used to establish a bound on the loss from using two levels of  $\alpha$ , provided the

distribution of  $\alpha^2(x)$  and  $\frac{1}{2} - \frac{1 - F(x)}{x f(x)}$  with respect to the density  $g$  satisfy the hazard rate

conditions. Here,  $\alpha(x) = \left( \frac{1 - F(x)}{x f(x)} \right)^{-1}$ , which maximizes the principal's payoff, is the

appropriate  $\alpha$ , provided it is increasing, which is guaranteed by a conventional hazard rate assumption. Note that  $\alpha$  can be taken as an exogenous function for the purposes of establishing a bound, although of course it is endogenous to the problem as a whole.

## List of Symbols

$\alpha$  alpha

$\beta$  beta

$\pi$  or  $\pi$  pi

$\geq$  greater than or equal to

$\infty$  infinity

$\rightarrow$  converges to

$\in$  is a member of

$\varepsilon$  epsilon

$\leq$  less than or equal to

$u_1$   $u$  sub one

1 one

l el

$\psi$  or  $\psi$  psi

$\mu$  mu

$\times$  times

x ex

Phi, null, sigma not used.

$\sum$  sum

---

<sup>1</sup> I thank Glenn Ellison and three anonymous referees for valuable comments that improved the paper.

<sup>2</sup> Lu and McAfee (1996) analyze the same model.

<sup>3</sup> The subscript refers to the number of cutoff points in the prioritization.

<sup>4</sup> Each consumer consumes one unit, although this turns out to be without loss of generality.

<sup>5</sup> Service probabilities are in the same units because the expected value to the consumer is the product of the probability and the value.

<sup>6</sup> It is a convenience to force  $H(N)=1$ , but larger quantities can be accommodated by permitting a mass point at  $N$ .

<sup>7</sup> The conditions have a special formulation in the case of service probabilities. It can readily be shown that the conditions on  $G$  reduce to  $qh(q)$  increasing and  $(N-q)h(q)$  decreasing in  $q$ , where  $h$  is the density of total supply.

<sup>8</sup> This inequality is a direct consequence of Milgrom and Weber's 1982 analysis of affiliation, because increasing functions of affiliated random variables are affiliated. A direct proof is provided in the Appendix A.

<sup>9</sup> This application has been explored by Rogerson (2001) for the case of uniform distributions. It is not an immediate application of the theorem because one of the variables is optimized to match the other. An example is sketched in Appendix B.