Auctioning Entry into Tournaments

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A research tournament model with heterogeneous contestants is presented. For a large class of contests the optimal number of competitors is two. This insight makes designing the tournament easier and highlights the importance of selecting highly qualified contestants. While customary uniform-price and discriminatory-price auctions are intuitively appealing mechanisms for solving this adverse selection problem, in practice they generally will not be efficient mechanisms for selecting contestants. Instead, we propose an alternative auction format that is equally simple to implement and efficiently selects the most qualified contestants to compete, regardless of the form of contestant heterogeneity.

I. Introduction

Tournaments have had a significant impact on the world's economic development. For example, in 1829 a research tournament sponsored by the Liverpool and Manchester Railway launched the world into the golden age of steam locomotion.[1] Offering a winner's prize of £500, the contest was designed to select an engine for the first-ever passenger line between two British cities. The stiffest competi-

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[1] We thank Anders Ekberg of Sweden's Chalmers University of Technology for bringing the details of this historic contest to our attention. See Day (1971) for details about the evolution of steam locomotives.

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tion fell between the *Rocket*, built by British inventors George and Robert Stephenson, and the *Novelty*, built by the Swedish inventor John Ericsson. The *Novelty* set a speed record of 51.5 kilometers per hour, but the *Rocket* ultimately won the contest with a top speed of 46 kilometers per hour after the *Novelty*'s engine collapsed. In the competition, the two engines' performances were so impressive that they convinced railway directors of the practicality of steam power and spawned an explosion of innovation in locomotives that spread across Britain and spilled into America.[2]

Tournaments continue to draw extensive financial resources today. In 1991, Lockheed's YF-22 defeated Northrop's YF-23 in the U.S. Air Force's advanced tactical fighter prototype "fly-off" competition, securing a long-term production contract estimated at the time to be worth more than $90 billion (Easterbrook 1991; Schwartz et al. 1991). More recently, in 1993, Whirlpool won a $30 million winner-take-all contest sponsored by two dozen utilities to build America's most energy-efficient refrigerator (Langreth 1994). These kinds of tournaments are effective because they mitigate the moral hazard problems that plague traditional contracts.

Conventional contracts are typically based on an agent's *absolute* level of performance; therefore, they may require significant monitoring efforts by the principal and can be expensive or impossible to enforce. In contrast, tournament prizes are based on an agent's performance *relative* to that of other competitors, which eliminates the need for the principal to invest in expensive monitoring and reduces court enforcement to simply verifying prize payment.

Labor tournaments, which have been analyzed extensively in the economics literature, are routinely employed by businesses to award bonuses and promotions when directly monitoring worker inputs is too costly to be worthwhile.[3] Research tournaments, like those mentioned above, are organized to stimulate the development of new products. The U.S. government is so enamored with procurement contests that national law requires all federal agencies to conduct competitive procurements whenever practical (United States 1987, sec. 253). This infatuation does have merit, for research tourna-

[2] It is interesting to note that Ericsson later immigrated to America and designed the *Monitor*, the world's first armored ship with a revolving gun turret. On March 9, 1862, the Union's *Monitor* fought the Confederate *Merrimack* in the world's first ironclad naval battle. See White (1960) and Davis (1994) for details.

[3] Alternately, monitoring may be easy, but the appropriate level of effort may be unknown because of uncertainty about the environment. For example, offering an effective sales commission requires knowing the expected number of sales, which may be highly uncertain. In contrast, offering a bonus to the most successful agent in conjunction with a base salary limits the financial exposure of the firm and ensures a reasonable level of effort, whether the product is popular or unpopular.

ments have proved to be cost-effective mechanisms for inducing innovation. For example, in the high-stakes international competition to invent the best technology for high-definition television (HDTV), both Europe and Japan invested hundreds of millions of dollars worth of their taxpayers' money to fund cooperative research efforts for HDTV development. In the United States, the Federal Communications Commission (FCC) merely sponsored a research tournament, using no taxpayer dollars, in which the "prize" was the right to license the new technology standard. By May 1993, editors of the *Economist* declared that "the FCC's decision to choose an HDTV standard through a neutral technical competition run and paid for by equipment makers, broadcasters and programme makers has been vindicated. The competition looks poised to produce a digital HDTV system years before anyone thought possible, immediately making obsolete the analogue systems promoted with lavish subsidies and much arm-twisting by government officials in Europe and Japan" ("HDTV: All Together Now," p. 74; see also Lewyn, Therrien, and Coy 1993).[4]

In contrast to the elaborate complexities of performance-based contracts, these tournaments are remarkably simple, requiring little information by the sponsor to be effective. If the sponsor offers a meaningful prize, which requires some information, and a clear explanation of the tournament's objective, then robust competition and a favorable outcome are virtually guaranteed if the competitors are competent. Therefore, the biggest problem a sponsor may face is selecting qualified contestants for the tournament.

Tournament sponsors go to great lengths and expend substantial resources trying to select the best-qualified contestants. Indeed, the primary purpose of holding multiple-round elimination tournaments in sports is to ensure that only the best-qualified competitors make it to the finals (see Rosen 1986). Other common mechanisms designed to reduce the effects of adverse selection include requiring credentials, accreditation, letters of recommendation, transcripts, tryouts, interviews, and examinations. Even in the realm of academia, one can argue that the arduous process of gaining tenure at a university is primarily designed to select the most qualified contestants (i.e., professors) to represent the institution in the ongoing competition of gaining grants and publishing papers. Once tenure is awarded, the presumption is that the qualifications and drive of the selected individual are sufficient to carry her successfully

[4] According to FCC Chairman Reed Hundt (1997), multiple digital HDTV signals will be broadcast to more than 50 percent of the U.S. market by the end of 1999, and analog broadcast signals will cease in the United States by 2006.

through the remaining rounds of the academic research tournament without the need for further supervision.

Foremost among the reasons for limiting entry into tournaments is the need to save on the cost of conducting and evaluating the competition. For example, academic journals often require submission fees for new papers in order to limit the cost of refereeing an excessive number of articles by "contestants" vying to get published. In some instances, evaluation costs may be high enough to shape the design of the entire competition. In one Department of Defense contract the federal government spent 182,000 work-hours to evaluate the proposals of just four firms (Fox 1974, p. 269).

A second reason for limiting entry into tournaments is to raise the level of contestant effort. In labor tournaments, Nalebuff and Stiglitz (1983) showed that allowing too many workers to compete for a prize may result in a reduction in the overall level of effort. Taylor (1995) proved an equivalent result for research tournaments with homogeneous contestants: allowing too many contestants reduces the total level of research effort and lowers the expected outcome of the tournament. With too many participants, contestants are discouraged from expending effort because their probability of winning becomes too small. Thus free and open entry is generally not the optimal participation policy, even if the cost of conducting and evaluating the contest is zero. Therefore, it is imperative to find an easy, inexpensive, and efficient method for restricting tournament entry to the best-qualified contestants.

This paper presents a model of research tournaments that allows for heterogeneous contestants and proves that for a large class of tournaments the optimal number of contestants is two. Besides reducing the costs of evaluating the tournament, restricting entry to two competitors optimizes the tournament outcome because it decreases the coordination problem of competing firms and minimizes the duplication of fixed costs. Additionally, when firms differ in their costs of conducting research, restricting entry to the two lowest-cost contestants enhances the competition because low-cost participants also conduct more research. Therefore, since the sponsor optimally restricts entry to just two competitors, limiting entry to the best-qualified contestants is one of the most important problems a tournament sponsor faces.

If the sponsor does not care how many contestants participate in her tournament, she can prevent the entry of many of the weakest contestants by charging an advertised entry fee. By announcing an entry fee, the sponsor deters the entry of any contestant whose expected profit from entry is less than the cost of the advertised fee. However, using an entry fee to restrict entry can backfire on the sponsor if an excessive number of firms gain entry or, worse yet, if only one firm enters. As previously mentioned, having too many contestants reduces the overall level of effort in a tournament, and, obviously, with only one contestant there is no competition and no research effort at all. Furthermore, calculating the exact entry fee needed to induce a specific number of competitors to enter in equilibrium would generally require the sponsor to know too much information to make hosting a tournament practical.

When the sponsor has some a priori knowledge of the optimal number of contestants, which we show is often two, a better alternative to using an advertised entry fee to restrict entry would be to host an entry auction. Like tournaments, auctions are easy to implement and have low informational requirements. Auctions can be easily structured to select a specific number of competitors, and the money collected during the entry auction can be used by the sponsor to offset the cost of the tournament winner's prize. Finally, since better-qualified contestants also have higher probabilities of winning the tournament prize, one would expect the best-qualified contestants to bid the most to gain entry into the tournament. Thus entry auctions are potentially ideal mechanisms for restricting tournament entry when sponsors have limited knowledge of contestant abilities but have an a priori understanding of the optimal number of tournament contestants.

Despite their familiar appeal, we show that customary uniform-price and discriminatory-price auctions are generally inefficient mechanisms for selecting the best-qualified contestants.[5] A better way to prevent the adverse selection of poor contestants is to use an alternative format we call the "contestant selection auction." The contestant selection auction is a variant of the all-pay auction commonly described in the literature on rent seeking (see, e.g., Baye, Kovenock, and De Vries 1993). This auction requires no additional information by the sponsor, is simple to implement, generates revenues equivalent to the uniform-price auction in the "independent private values" case, and is efficient at selecting the best-qualified contestants, regardless of the form of heterogeneity.

We begin our analysis in Section II by introducing a model of research tournaments that allows for heterogeneous contestants, provides closed forms, and is suitable for embedding in bidding environments. We provide sufficient conditions for the optimal number of contestants in this research tournament to be two and character-

[5] A uniform-price auction corresponds in the multiunit case to the Vickrey, or second-price, auction. In the uniform-price auction, all successful bidders pay the highest unsuccessful bid. In a discriminatory-price auction, all successful bidders pay their own bids. Therefore, a discriminatory-price auction is the multiunit equivalent of the first-price auction. For a general survey of auctions, see McAfee and McMillan (1987).

ize the conditions under which a uniform-price auction will be efficient in this contest for selecting the lowest-cost competitors. In Section III, we generalize the model and offer a theorem that provides sufficient conditions for environments under which customary auctions will fail to efficiently select the best-qualified contestants. For some types of heterogeneity, standard auctions fail to be efficient all of the time. In Section IV, we describe a variant of the all-pay auction that selects the best-qualified contestants, in equilibrium, without requiring any additional informational requirements beyond setting the appropriate prize.

II. A Research Tournament Model with Heterogeneous Contestants

Consider a research tournament in which there are $i = 1, 2, \ldots,$ n risk-neutral firms willing to compete for the prize, P. Prior to the tournament, each firm is independently endowed with a cost of conducting research, c, which is drawn from the continuous distribution $H(c)$ with density $h(c)$. After entering the contest, each firm i chooses a level of effort, z_i, at a total cost of $\gamma + c_i z_i$. The fixed cost γ is avoided if $z_i = 0$ is chosen. The choice of effort z_i results in a random innovation x, given by the cumulative distribution function $F^{z_i}(x)$, where F has support $[0, \bar{x}]$. Innovations are independently distributed across firms. The function F is assumed to have density f, so the density of i's innovation is $z_i F^{z_i - 1}(x) f(x)$. This probability structure would arise if the firm took z_i identical, independent draws from $F(x)$, where, for convenience, we do not restrict z_i to be an integer. The timing of the tournament is such that first each of the n firms simultaneously chooses whether to participate. After this decision, the set of competitors and their costs become common knowledge. Then each participating firm simultaneously makes its choice of z_i. Therefore, if J firms participate in the tournament and each starts with an initially worthless innovation ($x = 0$), the expected profit per firm without fixed costs is

$$\pi_i = P \int_0^{\bar{x}} \left[\prod_{j \neq i} F^{z_j}(x) \right] z_i F^{z_i - 1}(x) f(x)\, dx - c_i z_i = \frac{P z_i}{\sum_j z_j} - c_i z_i, \quad (1)$$

$$\frac{\partial \pi_i}{\partial z_i} = \frac{P \sum_{j \neq i} z_j}{\left(\sum_j z_j \right)^2} - c_i. \quad (2)$$

Since the expected profit function, π_i, is concave, the first-order condition (2) completely characterizes each competing firm's choice of z_i as follows: if $0 \geq (\partial\pi_i/\partial z_i)|_{z_i=0}$, firm i chooses $z_i = 0$; alternatively, if firm i chooses $z_i > 0$, then firm i's choice will follow the first-order condition.

THEOREM 1. Given a set of firms, there is a unique equilibrium in the subgame that involves positive z_i for the lowest-cost firms. (All proofs are in the Appendix.)

As part of theorem 1, we obtain the following expressions when M is the set of firms that choose positive effort levels and $|M|$ is defined to be the number of firms in M:

$$z_i = \frac{P(|M| - 1)}{\sum_{j\in M} c_j}\left[1 - \frac{c_i(|M| - 1)}{\sum_{j\in M} c_j}\right] \tag{3}$$

and

$$\pi_i = P\left[1 - \frac{c_i(|M| - 1)}{\sum_{j\in M} c_j}\right]^2. \tag{4}$$

From these expressions it is clear that a larger prize, P, will induce firms to conduct a greater amount of equilibrium effort, as well as raising the firms' expected profits. A subgame-perfect equilibrium for this research tournament arises when two conditions are satisfied: (1) the firms that pay γ make nonnegative profits and (2) the firms that do not pay γ would make negative profits by paying γ. Not surprisingly, there is some ambiguity in the equilibrium. In particular, if two firms have costs c_i and c_k that are sufficiently close and it is not profitable for both to enter and conduct research with positive z, then either one may enter and conduct research. However, the costs must be quite close, on the order of $|M|^{-2}$, as we show in the next result.

LEMMA 1. Suppose that M represents an entry equilibrium, and $k \in M$, $i \notin M$. Let $m = |M|$. Then

$$c_i \geq \frac{m^2 - m}{m^2 - m + 1}c_k.$$

According to lemma 1, either any equilibrium will involve the efficient firms or, when a low-cost firm is excluded, the firm's cost will be nearly that of the firms that enter. The bound is exact in the following sense. Suppose that firm $i \geq 2$ has cost c, and firm 1 has cost $[(m^2 - m)/(m^2 - m + 1)]c$. Choose γ so that the positive research of firms $\{2, \ldots, m + 1\}$ gives zero profits. Then firm 1 is just indifferent to conducting a positive amount of research; any higher cost and firm 1 is unwilling to research. Thus the bound in lemma 1 binds exactly in this extreme case. In what follows, we focus only on the efficient equilibrium, in which the first (lowest-cost) m firms enter and conduct research.

THEOREM 2. There is a unique number of firms, m, that is efficient for entry into the tournament and entry of the m lowest-cost firms is an equilibrium.

A consequence of theorem 1 (in particular, eq. [4]) is that, if m firms enter in equilibrium, then

$$\frac{(m-1)c_m}{\sum_{j=1}^{m} c_j} < 1. \tag{5}$$

Were inequality (5) to fail, the mth firm would earn negative profits in the subgame following entry. Thus inequality (5) is necessary, though not sufficient, for nonnegative profits. In what follows, we shall denote by $\overline{m}$ the largest m satisfying (5).

The sponsor's payoff is the realized best value of x, which has a cumulative distribution of $F(x)^{\sum_{i=1}^{m} z_i}$. Thus the sponsor's payoff depends only on $Z = \sum_{i=1}^{m} z_i$ and the payments made to the firms, which permits us to investigate the optimal number of participants by considering the cost of implementing an arbitrary Z. Some insight into the optimal number of participants is available from the symmetric case, where $c_i = c$ for all i. Each firm's profit in this symmetric case is equal to (4):

$$\pi_i = P\left[1 - \frac{c_i(m-1)}{\sum_{j \in M} c_j}\right]^2 - \gamma = \frac{P}{m^2} - \gamma.$$

By charging an entry fee, E, the sponsor can select the number of contestants, as well as recover the firms' profits. To induce entry of m firms, the sponsor sets an entry fee of $E = (P/m^2) - \gamma$; then the sponsor can obtain any desired total effort level Z by setting the prize equal to $P = mcZ/(m-1)$, provided that $m \geq 2$.[6] The total cost of procurement then is $P - mE = cZ + m\gamma$, which is minimized at $m = 2$.

[6] Note that at $m = 1$ no effort is expended because the sole entering firm faces no competition.

That two firms minimize the total cost is intuitive for, in the symmetric case, profits are zero; so the sponsor pays only the cost of effort. Given that the fixed costs γ induce economies of scale, the total cost is increasing in the number of contestants. This result generalizes to the case of firms with different costs as well. However, when firms have different costs, $m = 2$ is optimal, regardless of the presence of fixed costs and economies of scale, although a technical condition is required.

Define $\Delta_m = mc_m / (\sum_{j=1}^{m} c_j)$, where c_m is the cost of the mth-lowest-cost firm.

THEOREM 3. If Δ_m is nondecreasing, then the total procurement cost of obtaining a fixed level Z is minimized at $m = 2$.

The term Δ_m is just the highest marginal cost of research of m divided by the average marginal cost of research, and thus the condition that Δ_m be increasing says that the marginal cost is increasing relative to the average. The condition that Δ_m be nondecreasing is satisfied for a variety of examples. If c_i is constant, then Δ_m is nondecreasing. If there are constant increments to cost, $c_i = a + b_i$, or proportional increments to cost, $c_i = \alpha(1 + \delta)^i$, then Δ_m is nondecreasing. The next lemma provides a sufficient condition for Δ_m to be nondecreasing.

LEMMA 2. Suppose that $m \leq \overline{m}$ and

$$\frac{c_m}{c_{m+1}} \leq \frac{1}{m} + \frac{m-1}{m}\frac{c_m - 1}{c_m}.$$

Then $\Delta_m \leq \Delta_{m+1}$.

It is readily verified that both the constant increment and the proportional increment costs satisfy lemma 2. Thus, for a large class of research contests, the sponsor optimizes by setting $m = 2$ and choosing the prize and entry fees to implement zero profits and the desired total effort Z. By restricting entry to the two lowest-cost firms, tournament sponsors not only reduce their costs of evaluating the tournament but also induce a greater level of cumulative research effort since low-cost firms conduct more research per prize dollar. When all firms are identical, the sponsor can implement this optimal tournament even when she does not know γ by letting firms bid for the right to enter the tournament. This strategy implements the optimal tournament with homogeneous firms because the firms bid the efficient entry fee in a uniform-price auction. Practically speaking, however, one cannot expect the sponsor to have full information or firms to be identical. Therefore, one of the sponsor's primary problems in creating a meaningful contest is to identify the lowest-cost competitors for entry into the tournament.

Consider heterogeneity in the marginal cost of research, c_i, with c_i privately observed by the firm. Given entry, the expected profit of contestant i was previously shown to be (4), which is strictly decreasing in c_i. Therefore, it seems reasonable to believe that low-cost contestants would be willing to pay more to gain entry into the tournament than high-cost contestants, suggesting that a routine uniform-price auction would solve the adverse selection problem.

To simplify the analysis, consider the independent private values case and suppose that, after entry, the costs of the firms become common knowledge. Since research costs are private information prior to entry, potential contestants have to calculate their bids on the basis of expectations about the costs of other entrants. If the entry mechanism is efficient (i.e., if it selects the lowest-cost contestants), contestants will know that all other entrants will have lower costs than the "marginal" $m + 1$st bidder, who just misses gaining entry into the tournament. Therefore, if contestant i gains entry, it must be true that the marginal bidder had costs at least as large as c_i. Fix a firm i, defining c_i to be its cost of research. In an efficient equilibrium, this firm enters the tournament via the entry auction if c_i is less than the mth-highest cost of the other $n - 1$ firms. If we define c_m to be the cost of the marginal bidder (i.e., the mth-lowest cost of the remaining $n - 1$ firms), then contestant i's expected profit, without the cost of his bid, is[7]

$$\int_{c_i}^{\infty}\left\{\int_0^{c_m}\cdots\int_0^{c_m} P\left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2 \times \frac{h(c_1)}{H(c_m)}\frac{h(c_2)}{H(c_m)}\cdots\frac{h(c_{m-1})}{H(c_m)}\,dc_1\,dc_2\cdots dc_{m-1}\right\} h_{m:n-1}(c_m)\,dc_m.$$

Note that the expected profit given entry of contestant i, as shown earlier in (4), remains embedded within this new expression. Suppose that the uniform-price entry auction has a symmetric, pure-strategy equilibrium bidding function, $B(c)$. We consider the uniform-price auction to be an efficient selection mechanism in equilibrium if and only if it selects the most qualified (i.e., the lowest-

[7] To clarify our notation, we defined $h(c)$ to be the density of the distribution of marginal research costs, c. Here, $h_{m:n-1}(c_m)$ is defined to be the order statistic distribution of the cost of the marginal contestant, which is the mth-*lowest* cost out of the remaining $n - 1$ contestants other than contestant i.

cost) contestants for entry into the tournament. Therefore, to be efficient, $B(c)$ must be strictly decreasing in contestant costs. If such an equilibrium bidding function exists, the ex ante pre-auction expected profit of arbitrary contestant i, who holds actual cost c_i but bids $B(\hat{c})$, is

$$\Pi(\hat{c}, c_i) = \int_{\hat{c}}^{\infty} \left\{ \int_0^{c_m} \cdots \int_0^{c_m} P \left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j} \right]^2 \times \frac{h(c_1)}{H(c_m)} \cdots \frac{h(c_{m-1})}{H(c_m)} \, dc_1 \cdots dc_{m-1} - B(c_m) \right\} h_{m:n-1}(c_m) \, dc_m. \tag{6}$$

Using (6) and a lemma by Guesnerie and Laffont (1984) (in the Appendix), one can show that the only possible pure-strategy, symmetric equilibrium bidding function for this uniform-price auction is

$$B(c_i) = P \int_0^{c_i} \cdots \int_0^{c_i} \left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j} \right]^2 \times \frac{h(c_1)}{H(c_i)} \cdots \frac{h(c_{m-1})}{H(c_i)} \, dc_1 \cdots dc_{m-1}. \tag{7}$$

This bidding function is efficient only if the following hazard rate condition is met.

LEMMA 3. If $ch(c)/H(c)$ is decreasing in c, then $B(c)$ is decreasing in c. Alternatively, if $ch(c)/H(c)$ is nondecreasing in c, then $B(c)$ is nondecreasing.

Although this hazard rate condition is met by many cost distributions, it is disconcerting knowing that the uniform-price auction is not always efficient. For example, there is no efficient equilibrium in which costs are uniformly distributed on any interval $[0, \bar{c}]$. Determining whether the cost distribution in any particular tournament meets this requirement significantly increases the information burden placed on a sponsor, confounding one of the primary reasons for holding an auction. Moreover, there are other common types of contestant heterogeneity that we shall investigate in the next section in which neither the uniform-price auction nor the discriminatory-price auction is efficient. Thus it seems imperative that we find a more consistent solution to the adverse selection problem.

III. A General Characterization of the Failure of Customary Auctions as Selection Mechanisms

The problem of identifying the best contestants for participation is a difficult issue in most tournament settings. Surveying the literature on labor tournaments, McLaughlin (1988, p. 248) noted, "The real problem of tournaments with heterogeneous contestants arises if the contestants' types cannot be identified. Since tournaments which are mixed ex post do not induce optimal effort and it is costly to induce self-sorting . . . , the outcome is not efficient."[8] Because the adverse selection problem is common to most tournaments, this section addresses the issue using a general model that captures a variety of tournaments and is robust enough to handle different types of heterogeneity.[9]

Assume once more that there are $i = 1, 2, \ldots, n$ firms willing to participate in a tournament and the sponsor wishes to select only the $m < n$ best-qualified contestants to compete. To restrict entry, the sponsor solicits bids from contestants in a pretournament auction, where only the m highest bidders are allowed to compete in the tournament. To capture contestant heterogeneity in this model, by assumption, all potential contestants are endowed prior to the tournament with some attribute $w \in [\underline{w}, \overline{w}]$. We assume that the values $[w_1, \ldots, w_n]$ are drawn from a symmetric distribution with a continuous density. The value of each contestant's quality attribute is private information when contestants submit their bids, but, as in Section II, we abstract away from strategic misrepresentation by assuming that these attributes become common knowledge after the sponsor selects the m entrants and prior to each entrant's choice of effort.

By definition, contestant quality is monotonically increasing in w, so sponsors prefer contestants with larger endowments of w over those with smaller endowments. Therefore, w could represent a variety of measures of quality. In a labor tournament, w might represent a worker's human capital, whereas in a research tournament, w could represent a firm's technological sophistication, the inverse of

[8] For further discussions of adverse selection and self-sorting in labor tournaments, see Lazear and Rosen (1981) and O'Keeffe, Viscusi, and Zeckhauser (1984).

[9] We thank Sherwin Rosen for suggesting that we look for a general solution to the adverse selection problem.

the firm's incremental cost of research effort ($w = 1/c$), or a mapping of several qualitative measures into the single attribute w. We define $\psi_i(w_i, w_m)$ to be the expected profit of contestant i, endowed with w_i, given entry, when the endowment of the marginal contestant (the mth largest of the remaining $n - 1$ contestants other than i) is w_m. Denote by $h_{m:n-1}(w_m | w_i)$ the density of the marginal contestant $w_{m:n-1}$ given that i's endowment is w_i, where, by the assumed symmetry, $h_{m:n-1}$ does not depend on i.

To clarify the notation of this general model, we can compare it with the model presented in Section II, where the general model profit expression $\psi_i(w_i, w_m)$ would correspond to

$$\psi_i(c_i, c_m) = \int_0^{c_m} \cdots \int_0^{c_m} P\left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2$$

$$\times \frac{h(c_1)}{H(c_m)} \cdots \frac{h(c_{m-1})}{H(c_m)}\, dc_1 \cdots dc_{m-1}.$$

Also, equation (6) in Section II could be expressed more compactly using our general model as follows:

$$\Pi(\hat{c}, c_i) = \int_{\hat{c}}^{\infty} [\psi(c_i, c_m) - B(c_m)]\, h_{m:n-1}(c_m)\, dc_m$$

and (7) could be written $B(c_i) = \psi(c_i, c_i)$. Clearly, the general model subsumes the heterogeneous cost model presented in Section II, yet the general model is also characterized broadly enough to be representative of other types of tournaments with different forms of contestant heterogeneity and profit functions. Therefore, in a tournament with a uniform-price entry auction, we can use the general model to express the pre-auction expected profit of arbitrary contestant i, holding endowment w_i, who bids as though he held endowment $\hat{w}$ as follows:[10]

$$\Pi(\hat{w}, w_i) = \int_{\underline{w}}^{\hat{w}} [\psi_i(w_i, w_m) - B(w_m)]\, h_{m:n-1}(w_m | w_i)\, dw_m. \qquad (8)$$

On the basis of (8), the only possible bidding equilibrium for the uniform-price entry auction is

[10] Since quality is assumed to be increasing in w, we would substitute w for the *inverse* of costs, c. Also, the dependence of $h_{m:n-1}$ on w_i was not shown in Sec. II because independence was assumed.

$$B(w_i) = \psi_i(w_i, w_i). \tag{9}$$

Equation (9) suggests that, in equilibrium, contestant i will submit a bid that assumes that his value of w is equal to that of the marginal contestant. Thus, in equilibrium, each contestant bids as though he will wind up being the weakest contestant to gain entry into the tournament.

This kind of equilibrium bidding is prevalent in many auctions. As Vickrey (1961, p. 28) explained in his seminal paper on auctions, bids reflect the expected valuation of the marginal or "first rejected" bidder. For example, in an ascending oral (English) auction, buyers stop bidding up the price of an object as soon as the person who places the second-highest value on the object drops out of the bidding. Thus the object inevitably sells at a price just above the valuation of the marginal buyer, which is precisely what (9) represents. Conventional wisdom also maintains that uniform-price auctions elicit "honest" responses from bidders. Again, (9) conforms to conventional wisdom because it equals the true expected profit of the bidder, given the assumption that he is the weakest (marginal) entrant.

Using the general model to calculate the bidding function in a discriminatory-price auction results in the following proposed bidding equilibrium:

$$B_1(w) = \int_{\underline{w}}^{w} \exp\left[-\int_{y}^{w} \frac{h_{m:n-1}(\alpha|\alpha)}{H_{m:n-1}(\alpha|\alpha)}\,d\alpha\right] \frac{h_{m:n-1}(y|y)}{H_{m:n-1}(y|y)}\psi(y, y)\,dy. \tag{10}$$

This equilibrium bidding function also squares nicely with the literature on discriminatory-price auctions. The integral in (10) reflects the bid shading, which is common to discriminatory-price bidding.[11] In addition, the profit term, $\psi(y, y)$, continues to reflect the bidding behavior of the "weakest entrant" alluded to in uniform-price bidding and more generally in Vickrey's theoretical analysis of auctions.

Surprisingly, however, while both (9) and (10) reflect commonly held notions about the proper bidding behavior in uniform- and discriminatory-price auctions, neither bidding function is generally an efficient bidding equilibrium in a tournament entry setting, as shown in theorem 4.

THEOREM 4. If there exists $\tilde{w} \in [\underline{w}, \overline{w}]$ such that, for all $w < \tilde{w}$, $\psi(w, w) \geq \psi(\tilde{w}, \tilde{w})$, a symmetric, pure-strategy bidding equilibrium does not exist for the discriminatory-price or uniform-price entry auction.

[11] For an explanation of discriminatory-price bid shading, see McAfee and McMillan (1987, p. 709).

Theorem 4 provides sufficient conditions for uniform-price auctions and discriminatory-price auctions to fail to be efficient mechanisms for selecting entrants. The key insight into this theorem is that, given entry into these customary auctions, a larger bid does not increase a contestant's chance of winning the prize, but a smaller bid may save the contestant money precisely when he does *not* want to enter the tournament because he truly is the weakest entrant. Customary auctions create a powerful incentive for contestants to drop their bids since weak entrants may be better off submitting a bid of zero and losing nothing than submitting a large bid and paying to gain entry, only to lose the prize.

To expound further on theorem 4, consider the expected profits, given entry, of the weakest tournament entrant, whom we suitably name *weak*. As shown in (9) and (10), if a bidding equilibrium exists in these customary auctions, all contestants will submit bids as though they will be the *weak* entrant following the auction. Since *weak* is the least qualified contestant that gains entry into the tournament, all other entrants must be randomly endowed with quality attributes, w, which are larger than *weak*'s endowment. Now consider what happens to *weak*'s expected profits if we simultaneously improve his endowment along with the endowments of all other entrants so that *weak* remains the worst entrant. If the profits of *weak* do not rise with this general improvement in endowments, the hypothesis of theorem 4 is satisfied, and the bidding functions proposed in (9) and (10) will not be monotonically increasing in w. In that case, neither the uniform-price auction *nor* the discriminatory-price auction will be efficient.[12]

For example, in Section II, w would represent the inverse of each contestant's costs, so improving the endowments of all contestants would imply lowering all contestants' research costs. Lowering all contestants' research costs has the direct effect of raising *weak*'s expected profits by reducing his cost of conducting research. However, lowering costs makes research cheaper for every contestant, which raises the expected value of the winning innovation and has the indirect effect of lowering the expected profits of *weak*. When the hazard rate condition shown earlier in lemma 3 is not met, the indirect effect dominates the direct effect, so theorem 4 is satisfied and customary auctions are inefficient.

Consider two more examples, which provide insight into theorem 4 and illustrate why uniform-price and discriminatory-price auctions are often inefficient mechanisms for selecting contestants.

[12] By "efficiency," we mean auctions with symmetric, pure-strategy bidding equilibria increasing in w.

Example 1: An Off-the-Shelf Competition without Research

Suppose that a sponsor merely wishes to procure the best product available from among all existing innovations, with no additional research allowed following entry into the tournament. This kind of contest is probably the most common of all procurement tournaments and is routinely used to purchase off-the-shelf equipment. To save on costs, the sponsor conducts an entry auction to limit the number of products she must evaluate. After the auction, she requires the m highest-bidding firms to pay their entry fee (equal to the m + 1st–highest bid in a uniform-price auction, or equal to their own bids in a discriminatory-price auction) and then carefully evaluates the m entrants' products and awards the prize (i.e., the procurement contract) to the entrant who submitted the best product. In this simple competition, heterogeneity is manifest in the differing qualities of the contestants' endowed innovations, represented by w.[13]

Consider the expected profits, given entry, of *weak* in this off-the-shelf contest. By definition, *weak* has the worst product of all entrants, so he cannot win the prize and his expected profits given entry are zero. From theorem 4, if one simultaneously raises *weak*'s endowment along with the endowment of all other entrants so that *weak* continues to have the worst product of all entrants, then *weak* still cannot win the contest and his expected profits given entry remain at zero. Thus the hypothesis of theorem 4 is satisfied and neither a uniform-price auction nor a discriminatory-price auction will be efficient.

Since *weak*'s expected profit equals zero regardless of the value of his endowment, this implies that $\psi(w, w) = 0$ for all endowments of w. Therefore, according to (9) and (10), the only possible symmetric, pure-strategy bidding equilibrium is $B(w) = 0$ for all values of w in both auctions.[14] Of course, it cannot possibly be an equilibrium for all contestants to bid zero because then any firm could increase its expected profits by submitting an infinitesimally small bid, guaranteeing entrance and at least a small chance of winning the prize. Therefore, in this simple off-the-shelf procurement contest, there is no pure-strategy, symmetric bidding equilibrium for

[13] This off-the-shelf contest corresponds to the tournament of Sec. II in which contestants are endowed with different innovations $x_i > 0$ prior to the tournament and research is not allowed after entry ($z_i = 0$).

[14] In the Appendix we confirm this analysis by deriving this bidding result explicitly.

either the uniform-price or the discriminatory-price auction.[15] Neither auction efficiently selects the best-qualified competitors, even in this most basic of competitions.[16]

Example 2: An Off-the-Shelf Competition with Research

For our second example illustrating theorem 4, consider an off-the-shelf competition in which contestants are allowed to conduct additional research following tournament entry. To avoid unnecessary complexity, let us assume that the marginal cost of conducting research after entry, c, is identical for all contestants. Define w_i to be the endowment of arbitrary entrant i and let $w_{\max} = \max_i\{w_i\}$.

Obviously, only the entrant holding $w_{\max}$ can win the tournament with his original endowment. All other entrants must engage in additional research following entry and draw an innovation better than $w_{\max}$ if they are to have any chance of winning. This difference in the probability of winning with one's endowment affects each contestant's probability of winning and their equilibrium effort-level strategies.

Applying the hypothesis of theorem 4, consider what happens to the profits of *weak* in this tournament when one simultaneously raises the endowments of all entrants. First, this action makes it more difficult for the weakest entrant to win because it raises the expected value of the innovation necessary to win the prize. This, in turn, raises the expected cost of drawing a winning innovation and lowers the expected profits of the weakest contestant. Moreover, if the endowment of the *best* entrant is such that $H(w_{\max}) \geq e^{-c/P}$, then it ceases to be worthwhile for any other contestant to try to win because the probability of obtaining a new innovation better than $w_{\max}$ is too slim to be worth the cost of the additional research effort. This effect is reflected in the following lemma.

LEMMA 4. If the value of the best entrant's starting innovation is such that $H(w_{\max}) \geq e^{-c/P}$, then none of the tournament contestants will conduct additional research following entry.

[15] Surprisingly, the uniform-price auction does not even have a symmetric, mixed-strategy bidding equilibrium in this contest (see Fullerton [1995] for details).

[16] There are asymmetric "bullying" equilibria for the uniform-price entry auction in which exactly m contestants submit bids of at least P and all other contestants submit bids of zero. In these equilibria, no fee is paid by the m entrants since the m + 1st–highest bid was zero. As for the contestants who do not gain entry, their profits are zero and cannot be improved since the only way they could gain entry would be to also bid P or higher. This would then make the m + 1st entry fee P or higher, leading to an expected loss for all entrants. Finally, these asymmetric bullying equilibria are not efficient since the identity of the m contestants bidding P is specified independently of initial qualifications.

If $c > 0$ and $P < \infty$, there will always be some interval of endowments at the upper end of the distribution of w, where the condition specified in lemma 4 is satisfied. Since contestants with these highly qualified endowments know that neither they nor their potential competitors will ever conduct additional research following entry, they assess the tournament just as though the competition was run without additional research. But, were they to be the weakest entrant, this implies that $\psi(w, w) = 0$ for these contestants, and therefore, there is still no efficient bidding equilibrium for the uniform-price or discriminatory-price auction even when research is allowed in an off-the-shelf competition.[17] Since almost any research competition would include competitors starting with different off-the-shelf endowments, this example indicates that customary auctions will rarely be efficient mechanisms for selecting contestants.

IV. A General Solution to the Adverse Selection Problem in Tournaments

Since customary auctions are generally inefficient mechanisms for selecting contestants, in this section we propose an alternative auction format that is efficient, regardless of the form of contestant heterogeneity. This new auction, which we call a "contestant selection auction," is implemented as follows: First, the tournament sponsor conducts an all-pay auction in which *all* bidders must pay their own bids regardless of whether they gain entry into the tournament. Following the auction, the m highest-bidding contestants are allowed to enter the tournament and are also given an interim prize, K, for becoming "finalists" in the competition. Finally, these m entrants compete for the primary tournament prize, P, with the sponsor paying the prize to the finalist that wins the overall competition.

In the contestant selection auction, if arbitrary contestant i, holding endowment w_i, bid as though he held $\hat{w}$, his pre-auction expected profit would be

$$\Pi(\hat{w}, w_i) = \int_{\underline{w}}^{\hat{w}} \psi(w_i, y) h_{m:n-1}(y | w_i) dy + K \int_{\underline{w}}^{\hat{w}} h_{m:n-1}(y | w_i) dy - B(\hat{w}). \tag{11}$$

On the basis of (11), the unique candidate for a symmetric, pure-strategy equilibrium bidding function for this contestant selection auction is

[17] See the Appendix for an explicit mathematical derivation of this result.

$$B(w) = \int_{\underline{w}}^{w} \psi(y, y) h_{m:n-1}(y|y)\, dy + K \int_{\underline{w}}^{w} h_{m:n-1}(y|y)\, dy. \tag{12}$$

A sufficient condition for (12) to be strictly increasing in w is $\psi(w, w) \geq 0$ for all w. Since tournament entrants would never undertake a research strategy that gives them *negative* expected profits following entry, this condition is always true. Though (12) is increasing in w, showing that it is an equilibrium bidding function is difficult in the general case because the lemma by Guesnerie and Laffont (1984) also requires

$$\frac{\partial[\psi(w, \hat{w}) + K]\, h_{m:n-1}(\hat{w}|w)}{\partial w} \geq 0.$$

However, this is clearly satisfied for independence of w_i and w_m, which leads us to theorem 5.

THEOREM 5. With independent types, the contestant selection auction is an efficient mechanism for selecting the best-qualified contestants to participate in a tournament.

The intuition for theorem 5 is straightforward given our earlier explanation of why tournament auctions fail. By making *all* contestants pay their bids, and not just those who gain entry, the sponsor creates a situation in which entry is always weakly preferred over nonentry for every contestant. Since contestants must pay their bids even if they do not gain entry, they are never made better off by shunning entry. Therefore, the contestant selection auction sidesteps the problem in which contestants reduce their bids to avoid becoming the marginal entrant. This all-pay format alone is enough to induce a bidding function that is weakly monotonic in w. But efficiency requires a bidding function that is strictly monotonic in w, making tournament entry strictly preferred over nonentry even for the marginal contestant. This is achieved by offering the m highest bidders the interim prize, K.

To illustrate this new bidding equilibrium, consider again the off-the-shelf procurement contest of example 1. This case is perhaps the most difficult for an entry auction since in that tournament $\psi(w, w) = 0$ for all w. However, if the sponsor uses a contestant selection auction in this off-the-shelf contest, the equilibrium bidding function will be efficient because it is strictly increasing in w:

$$B(w) = K \int_{\underline{w}}^{w} h_{m:n-1}(y|y)\, dy.$$

The contestant selection auction is simple to implement and does not place a significant information burden on the sponsor because the total cost of conducting the tournament is independent of the size of the interim prize, K. No information on the distribution of attributes or types is required, and K could theoretically be as small as a dollar, although in practice it should probably be large enough for the contestants to consider it worthwhile to win. Moreover, the contestant selection auction generates the same bidding revenue for the sponsor as an efficient uniform-price auction, which is the subject of theorem 6.

THEOREM 6. With independent types, the expected cost of implementing a tournament using the contestant selection auction is independent of the interim prize, K, and equivalent to the expected cost of implementing a tournament using an efficient uniform-price entry auction.

Revenue equivalence theorems are common in much of auction theory for independent valuations. In the contestant selection auction, when $K = 0$ (i.e., a standard all-pay auction), contestants will individually bid less than in an efficient uniform-price auction, but the sponsor collects the same total revenue because all contestants must pay their bids and not just those who gain entry into the competition. When $K > 0$, the equilibrium expected total revenue collected from the bids is exactly mK greater than the revenue collected in the uniform-price auction or the $K = 0$ auction, thereby offsetting the interim prize payments to the m entrants. The contestant selection auction reduces the problem of designing a tournament to one of choosing an appropriate primary prize, P, and number of contestants, m, which we have shown will often be two. Therefore, the advantages of the contestant selection mechanism over other means of choosing contestants are that (*a*) the auction is theoretically efficient for all types of heterogeneity; (*b*) the cost of the auction is independent of the sponsor's choice of the interim prize, K; and (*c*) the auction generates expected revenues equivalent to an efficient uniform-price auction. These characteristics make the contestant selection auction an ideal solution to the adverse selection problem in many tournaments.

V. Conclusion

Tournaments represent a significant, and increasingly popular, allocation mechanism in modern economies. In the last decade, for instance, even China has switched to a research system that is increasingly reliant on competition.[18] A major advantage of tournaments

[18] Not only is the new Chinese competition paying off in better research outcomes, but the policy is also making a sociological impact on the nation by attracting many young scientists and engineers back to China from the United States and Europe ("Science in China," 1995).

over commissions and other pay-for-performance methods is that tournaments require less information about contestants and the environment. As we have shown, in a wide class of environments, the tournament is optimally restricted to just two contestants. This finding is particularly relevant to the many industries that have been downsizing in recent years because of mergers. For example, the Department of Defense procurement budget has fallen from a high of $134 billion in 1985 during the Reagan buildup to $43 billion in 1996.[19] This post–Cold War decline in defense procurement spending has led analysts to predict that 80 of the top 100 defense firms will quit the industry by the turn of the century ("Survey of Military Aerospace," 1994). Yet, if effective research competition requires only two contestants, then future Defense acquisition efforts may not necessarily be jeopardized by downsizing as long as key technological rivalries remain intact.[20]

Restricting the number of contestants in a tournament reduces the costs of evaluating the competition, minimizes the duplication of fixed costs, and eliminates the coordination problem, wherein one contestant continues to exert effort after another has successfully innovated. Restricting the number of contestants may also increase the total effort exerted. However, with heterogeneous contestants, it is important to select the best-qualified contestants when the sponsor restricts entry.

Appropriate entry fees can, in principle, restrict participation to the best-qualified contestants. However, setting entry fees requires detailed knowledge of the environment and potential contestants, knowledge that will not be available in practice. Moreover, poorly set entry fees may lead to no contestants, or a huge number, and since the optimal entry fee varies with the specific realization of the environment's uncertainty, entry fees will not generally be robust to a lack of knowledge of the environment. (In contrast, poorly set prizes may cause inefficient levels of effort, but the effort will be a continuous function of the prize, whereas a poorly set entry fee may lead to no contestants and zero effort.) Thus using entry fees violates the spirit of tournaments, which is to get good outcomes with low informational requirements.

Auctions offer a promising alternative means of setting entry fees because auctions permit the entry fee to reflect information held by

[19] Procurement figures are in constant 1997 dollars. Source: U.S. Department of Defense (1996).

[20] Kovacic and Smallwood (1994) contend that the greatest benefit to competition in the defense industry is its power to spur ingenuity and offer a compelling argument for the need to preserve key technological rivalries in the firms that do survive the industry shakeout.

the contestants but not available to the sponsor. However, the case for efficiency, which is so strong in single-item auctions, is much weaker in tournaments, for with the two standard multiunit auctions, efficiency will not generally arise as an equilibrium when the bidders vary in their initial product endowments. Even when bidders have identical product endowments but vary in their costs of conducting research, efficient selection requires a hazard rate condition. This result is quite intuitive: auction prices reflect the value of entering for the marginal entrant, and in the tournament context, the marginal entrant is the worst entrant and would like to be excluded.

In contrast, the all-pay auction, amended to provide a nominal prize to successful bidders, selects the best contestants in a large class of tournament environments. It does this by eliminating the desire of poorly qualified contestants to be excluded (since one pays whether one is included or excluded) and by inducing a desire to be included (because of the interim prize). This auction, which we call the contestant selection auction, has minimal informational requirements. The sponsor needs only to set the prize for the auction and choose the number of participants. Setting the prize appropriately requires some knowledge of the environment, but behavior is continuous in the prize, and therefore the mechanism is robust to small errors. In many environments, the optimal number of contestants is two, and it may be possible to know this without further detailed knowledge of the environment. No further knowledge is needed to implement the contestant selection auction. Even the interim prize is irrelevant; in theory it could be a penny. In practice, it probably needs to be a sufficiently large amount that contestants will consider it.

One possible use for the contestant selection auction is as a cost-saving alternative to the Competition in Contracting Act (CICA) of 1984. The act requires all federal agencies to conduct "free and open competition" whenever practical in an effort to ensure that no contestant is ever "unfairly" excluded from a procurement.[21] The results of this act have been predictable, and in one procurement competition to buy \$11,000 worth of equipment, the Army spent \$5,000 reproducing and sending out invitations to more than 100 firms, which responded to the "free and open competition" advertisement (Gansler 1989, p. 182). Had an auction been used to restrict entry, the Army would have saved the \$5,000 it spent reproducing invitations, made additional revenue from the auction's bids,

[21] See Gansler (1989, p. 182) for a discussion of the effects of CICA (Public Law 98-92) on defense contracting.

and still have received a quality product. Moreover, if the government were to limit entry by using a contestant selection auction, the only firms that should ultimately be excluded from the competition are those that would normally not have won the contest. By requiring firms to pass through an entry auction prior to competing, the government can legitimately limit the number of competitors while maintaining its sense of "fairness" to all firms.

The contestant selection auction also has applications to competitions other than research tournaments. For example, architectural competitions are frequently held to pick designs for prominent structures.[22] In these contests, potential competitors do not submit monetary bids, but rather they submit design proposals, of which a small number are selected for the final competition. In terms of selection efficiency, it does not matter whether a bid is in dollars or in effort. The important thing is that the bid is sunk at the time the entry decision is made. Recognition as a "finalist" in an elite competition may itself serve as a suitable interim prize, and, of course, the grand prize is having one's design selected for the structure.

Finally, the labor economics literature suggests that individuals pursue higher education not only to gain human capital but also to signal employers that they are more talented job contestants (Spence 1973; Weiss 1995). Thus, if one corrects for bias in family wealth, paying for a costly education may be viewed as a high-cost bid to gain entry into a restrictive labor tournament. Insofar as all college students must pay to attend school, this mechanism resembles an all-pay auction. While the big prize for attending an Ivy League school is most likely a high-paying job following graduation, the "interim prize" to gaining entry into an exclusive school is presumably an exceptional education and perhaps some valuable contacts. Therefore, without too much of a stretch, college entrance competition can be seen to resemble a contestant selection auction, suggesting that employers are justified in hiring graduates of expensive private schools because they actually may be the best-qualified employee candidates.

Appendix

We offer sketches of each proof here. Complete proofs are in a technical appendix available from the authors on request. The following lemma is

[22] A member of the audience at an economic meeting in Mannheim suggested this application.

used repeatedly in the paper. This lemma was first proved in the generality shown here by Guesnerie and Laffont (1984); however, special cases were used by several authors, notably Myerson (1981), prior to this. Subscripts denote partial derivatives.

LEMMA A1. Suppose that v: $[a, b] \to \mathbb{R}$ is twice continuously differentiable, and consider four assumptions: (1) For all r and for all x, $v(r, x) \leq v(x, x)$. (2) For all x, $v_1(r, x) = 0$ when evaluated at $r = x$. (3) For all x, $v_{12}(r, x) \geq 0$ when evaluated at $r = x$. (4) For all r and for all x, $v_{12}(r, x) \geq 0$. Then the following are true: assumption 1 implies assumptions 2 and 3; moreover, assumptions 2 and 4 together imply assumption 1.

Proof of Theorem 1

Order the set of firms from lowest-cost to highest-cost; for the purpose of this proof only, subscripts will refer to the ordered costs within the set of firms. An equilibrium is a subset of these firms M such that $z_i = 0$ for $i \notin M$ and $z_i > 0$ for $i \in M$. Therefore,

$$i \notin M \Rightarrow 0 \geq \left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} = \frac{P}{\sum_{j \in M} z_j} - c_i$$

or

$$c_i \geq \frac{P}{\sum_{j \in M} z_j};$$

$$i \in M \Rightarrow c_i = \frac{P \sum_{j \in M, j \neq i} z_j}{\left(\sum_{j \in M} z_j\right)^2}$$

and $z_i > 0$.

Let $|M|$ be the number of elements in M and sum the first-order conditions over $i \in M$. Then the requirement $z_i > 0$ yields the following: For $i \in M$:

$$c_i = \frac{P \sum_{j \in M, j \neq i} z_j}{\left(\sum_{j \in M} z_j\right)^2} < \frac{P \sum_{j \in M} z_j}{\left(\sum_{j \in M} z_j\right)^2} = \frac{\sum_{j \in M} c_j}{|M| - 1}.$$

For $i \notin M$:

$$c_i \geq \frac{P}{\sum_{j \in M} z_j} = \frac{\sum_{j \in M} c_j}{|M| - 1}.$$

Therefore, any equilibrium involves only the lowest-cost firms, say m of them, and is characterized by

$$c_m < \frac{\sum_{j=1}^{m} c_j}{m - 1} \leq c_{m+1},$$

which, by induction, is unique. Q.E.D.

Proof of Lemma 1

Suppose that $c_i < c_k = \max_{j \in M}\{c_j\}$, and $i \notin M$. Since $k \in M$, if $c_k \geq (c_i + \sum_{j \in M} c_j)/|M|$, then entry of i would cause $z_k = 0$, yielding profit for firm i of

$$\pi_i = P\left[1 - \frac{c_i(|M| - 1)}{\sum_{j \in M} c_j - c_k + c_i}\right]^2 > P\left[1 - \frac{c_k(|M| - 1)}{\sum_{j \in M} c_j}\right]^2 \geq \gamma;$$

but this implies $i \in M$, which contradicts our earlier supposition. Therefore, $c_k < (c_i + \sum_{j \in M} c_j)/|M|$.

Since $i \notin M$, letting $m = |M|$, we have

$$c_i m \geq c_k(m - 1)\left(1 + \frac{c_i}{\sum_{j \in M} c_j}\right) \geq c_k(m - 1)\left(1 + \frac{c_i}{mc_k}\right).$$

Multiply by mc_k to get $m^2 c_i c_k \geq c_k(m - 1)(mc_k + c_i)$ or $[m^2 - (m - 1)]c_i \geq m(m - 1)c_k$. Q.E.D.

Proof of Theorem 2

Note that

$$\gamma \geq P\left[1 - \frac{c_k(k - 1)}{\sum_{j \leq k} c_j}\right]^2 \quad \text{iff} \quad 1 - \sqrt{\frac{\gamma}{P}} \leq \frac{c_k(k - 1)}{\sum_{j \leq k} c_j}$$

implies

$$\left(1 - \sqrt{\frac{\gamma}{P}}\right)\sum_{j\le k} c_j \le c_k(k-1) \Rightarrow \left(1 - \sqrt{\frac{\gamma}{P}}\right)\sum_{j\le k+1} c_j$$

$$\le c_k(k-1) + \left(1 - \sqrt{\frac{\gamma}{P}}\right)c_{k+1} \le kc_{k+1},$$

which implies that

$$\gamma \ge P\left(1 - \frac{kc_{k+1}}{\sum\limits_{j\le k+1} c_j}\right)^2.$$

Thus, if it is unprofitable for firm k to enter, it is unprofitable for firm $k+1$ to enter, showing that the equilibrium m is uniquely determined. Q.E.D.

Proof of Theorem 3

Prize $P = (Z\sum_{j=1}^{m} c_j)/(m-1)$, and the optimal entry fee with m participants is

$$E = P\left[1 - \frac{c_m(m-1)}{\sum\limits_{j\in M} c_j}\right]^2 - \gamma,$$

which are the profits of the mth-highest firm. The total cost of procurement with m participants is

$$TC_m = P - mE = P\left\{1 - m\left[1 - \frac{c_m(m-1)}{\sum\limits_{j\in M} c_j}\right]^2\right\}$$

$$+ m\gamma = Z\sum_{j=1}^{m} c_j\left(-1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2\right) + m\gamma.$$

Note that $1 \le \Delta_m \le m/(m-1)$ since $(m-1)c_m/(\sum_{j=1}^{m} c_j) < 1$ (profits are nonnegative) and $c_j \le c_m$, for all $j < m$:

$$TC_{m+1} - TC_m = \gamma + Z\sum_{j=1}^{m+1} c_j\left(-1 + 2\Delta_{m+1} - \frac{m}{m+1}\Delta_{m+1}^2\right)$$

$$- Z\sum_{j=1}^{m} c_j\left(-1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2\right),$$

which is positive if $\Delta_{m+1} - \Delta_m \geq 0$. So $TC_{m+1} \geq TC_m$, and the optimum occurs at $m = 2$. Q.E.D.

Proof of Lemma 2 (by Induction)

First note that $c_2 \geq c_1 \Rightarrow \Delta_2 \geq \Delta_1$. Rewrite $\Delta_m \geq \Delta_{m-1}$ as

$$1 - \frac{(m-1)c_{m-1}}{mc_m} \geq \frac{c_m}{\sum_{j=1}^{m} c_j}.$$

By assumption,

$$\frac{1}{m} + \frac{m-1}{m}\frac{c_{m-1}}{c_m} \geq \frac{c_m}{c_{m+1}}.$$

Therefore,

$$1 - \frac{m-1}{m}\frac{c_{m-1}}{c_m} \geq \frac{c_m}{\sum_{j=1}^{m} c_j}$$

implies

$$\frac{m+1}{m} = 1 - \frac{(m-1)c_{m-1}}{mc_m} + \frac{1}{m} + \frac{(m-1)c_{m-1}}{mc_m} \geq \frac{c_m}{\sum_{j=1}^{m} c_j} + \frac{c_m}{c_{m+1}},$$

which implies $\Delta_{m+1} \geq \Delta_m$. Q.E.D.

Deviation of Equation (7)

Consider the uniform-price auction, and suppose that B is an equilibrium bidding function. A firm with cost c_i that bids $B(\hat{c})$ earns expected profits of $\Pi_i(\hat{c}, c_i)$, as defined in equation (6) in the text. Taking the partial with respect to $\hat{c}$ gives us

$$\frac{\partial \Pi_i}{\partial \hat{c}} =$$

$$-\left\{\int_0^{\hat{c}} \cdots \int_0^{\hat{c}} P\left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2 \frac{h(c_1)}{H(\hat{c})}\frac{h(c_2)}{H(\hat{c})} \cdots \frac{h(c_{m-1})}{H(\hat{c})} dc_1\, dc_2 \cdots dc_{m-1}\right\}$$
$$\times\, h_{m:n-1}(\hat{c}) + B(\hat{c})h_{m:n-1}(\hat{c}).$$

Since $c_i(m-1)/(c_i + \sum_{j=1}^{m-1} c_j)$ is increasing in c_i, this implies that $\partial^2\Pi_i/\partial\hat{c}\partial c_i \geq 0$. Thus, using theorem A1, we get equation (7) in the text.

Proof of Lemma 3

By integrating by parts, substituting, summing, and canceling, one can show that

$$\frac{\partial B(c)}{\partial c} = P\frac{(m-1)}{[H(c)]^{m-1}}\sum_{k=1}^{m-1}\int_0^c\cdots\int_0^c 2\left[1-\frac{c(m-1)}{c+\sum_{j=1}^{m-1}c_j}\right]$$

$$\times\left[\frac{H(c_k)}{\left(c+\sum_{j=1}^{m-1}c_j\right)^2}\right]\prod_{j\neq k}^{m-1}h(c_j)\left[\frac{ch(c)}{H(c)}-\frac{c_k h(c_k)}{H(c_k)}\right]dc_1\cdots dc_{m-1}.$$

Therefore, if $ch(c)/H(c)$ is *decreasing* in c, $\partial B(c)/\partial c < 0$. So a pure-strategy equilibrium exists that is efficient. On the other hand, if $ch(c)/H(c)$ is increasing, then any equilibrium is inefficient. Q.E.D.

Proof of Theorem 4

If the sponsor conducts a uniform-price entry auction and contestant i deviates from the bidding equilibrium by bidding as type $\hat{w}$, his ex ante expected profit from the tournament will be $\Pi(\hat{w}, w_i)$, defined in equation (8) in the text. From lemma A1, the first-order conditions for this expression suggest that the only possible equilibrium for this uniform-price entry auction is the bidding function $B(w_i) = \psi_i(w_i, w_i)$. This implies that firms must bid their expected profits given entry into the tournament, under the assumption that they will have the smallest endowment of all entrants. But the condition that there exists $\tilde{w} \in [\underline{w}, \overline{w}]$ such that, for all $w < \tilde{w}$, $\psi(w, w) \geq \psi(\tilde{w}, \tilde{w})$ implies that this bid is either decreasing for w sufficiently close to $\tilde{w}$ or constant for all $w < \tilde{w}$. Therefore, a symmetric, increasing, pure-strategy bidding equilibrium cannot exist.

In a discriminatory-price auction, label the bidding function as $B_1(w)$. Then a bidder who holds w_i but bids as though he holds $\hat{w}$ has expected profits of

$$\Pi(\hat{w}, w_i) = \int_{\underline{w}}^{\hat{w}}[\psi(w_i, w_m) - B_1(\hat{w})]h_{m:n-1}(w_m|w_i)\,dw_m.$$

The first-order conditions give

$$B_1'(w) = [\psi(w, w) - B_1(w)]\frac{h_{m:n-1}(w|w)}{H_{m:n-1}(w|w)}.$$

When this is combined with $B_1(\underline{w}) = 0$, which is a consequence of not accepting negative bids, we get equation (10) in the text. Provided that

$$\lim_{y\to 0}\int_y^w \frac{h_{m:n-1}(\alpha|\alpha)}{H_{m:n-1}(\alpha|\alpha)}\,d\alpha = \infty$$

(a sufficient condition is $h_{m:n-1}(\underline{w}|w) > \epsilon > 0$), we get

$$B_1'(w) =$$

$$\left\{\int_{\underline{w}}^{w} \exp\left[-\int_{y}^{w} \frac{h_{m:n-1}(\alpha|\alpha)}{H_{m:n-1}(\alpha|\alpha)}\, d\alpha\right] \frac{h_{m:n-1}(y|y)}{H_{m:n-1}(y|y)} [\psi(w, w) - \psi(y, y)]\, dy\right\}$$

$$\times \frac{h_{m:n-1}(w|w)}{H_{m:n-1}(w|w)}.$$

But the condition that there exists $\tilde{w} \in [\underline{w}, \overline{w}]$ such that, for all $w < \tilde{w}$, $\psi(w, w) \geq \psi(\tilde{w}, \tilde{w})$ implies that this bid is either decreasing for w sufficiently close to $\tilde{w}$ or constant for all $w < \tilde{w}$. Q.E.D.

Proof of Example 1

First, using an explicit representation of a uniform-price auction in the off-the-shelf contest, let w be independently distributed with cumulative distribution function $H(w)$ and density $h(w)$. There are m firms that gain entry, and let w_s be the smallest w of all entrants, implying that all entrants can be said to have been drawn independently from distribution $[H(w) - H(w_s)]/[1 - H(w_s)]$. Thus a firm holding w_i that bids as though it held r would expect a profit of

$$\pi(r, w_i) = P\int_{\underline{w}}^{\min\{r, w_i\}} \left[\frac{H(w_i) - H(w_m)}{1 - H(w_m)}\right]^{m-1} h_{m:n}(w_m)\, dw_m - \int_{\underline{w}}^{r} B(w_m)\, h_{m:n}(w_m)\, dw_m.$$

According to lemma A1 by Guesnerie and Laffont (1984), the first-order conditions for a maximum require $\pi_1(w_i, w_i) = 0$, which gives us $0 = -B(w_i)\, h_{m:n}(w_i)$. This implies that the only possible equilibrium bidding is $B(w) = \psi(w, w) = 0$ for all w, verifying our statement in the text and confirming that there is no uniform-price auction bidding equilibrium for this competition. (Note: The proof of no discriminatory-price auction is done in the same manner, with the same results.)

Proof of Lemma 4

For $i \neq \max$:

$$\pi_i = P\frac{z_i}{Z}[1 - H(w_{\max})^Z] - cz_i,$$

$$\left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} = P\frac{1 - H(w_{\max})^Z}{Z} - c,$$

and

$$\frac{\partial^2 \pi_i}{(\partial z_i)^2} \leq 0.$$

For i = max:

$$\pi_{\max} = PH(w_{\max})^Z + P\frac{z_{\max}}{Z}[1 - H(w_{\max})^Z] - cz_{\max},$$

$$\frac{\partial \pi_{\max}}{\partial z_{\max}} = P\frac{Z - z_{\max}}{Z^2}[ZH(w_{\max})^Z \log H(w_{\max}) + 1 - H(w_{\max})^Z] - c,$$

$$\frac{\partial^2 \pi_{\max}}{(\partial z_{\max})^2} \leq 0,$$

$$\left.\frac{\partial \pi_{\max}}{\partial z_{\max}}\right|_{z_m=0} = \frac{P}{Z}[1 - H(w_{\max})^Z + ZH(w_{\max})^Z \log H(w_{\max})] - c.$$

Therefore,

$$\left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} \leq 0 \Rightarrow \left.\frac{\partial \pi_{\max}}{\partial z_{\max}}\right|_{z_{\max}=0} \leq 0.$$

Thus, using L'Hôpital's rule, one can show that $Z = 0$ if and only if $H(w_{\max}) \geq e^{-c/P}$. Q.E.D.

When firms are allowed to conduct research after entry, lemma 4 tells us that $Z = 0$ if and only if $H(w_{\max}) \geq e^{-c/P}$. This implies that, for all tournaments in which $c > 0$ and $P < \infty$, $e^{-c/P} < 1$. So the firms with the best possible endowments (i.e., greater than this critical value of w) will *never* do additional research following entry. If the sponsor conducts a uniform-price entry auction, the expected profits of one of these firms holding w, which bids as though it holds r but never does additional research, is

$$\pi(r, w_i) = P\int_{\underline{w}}^{\min\{r, w_i\}}\left[\frac{H(w_i) - H(w_m)}{1 - H(w_m)}\right]^{m-1} h_{m:n}(w_m)\,dw_m - \int_{\underline{w}}^{r} B(w_m)\,h_{m:n}(w_m)\,dw_m.$$

Using lemma A1 and taking the derivative with respect to r, evaluated at $r = x$, leaves us with

$$\pi_1(r, w_i)|_{r=w_i} = 0 \Rightarrow B(w) = \psi(w, w) = 0.$$

This means that the only possible pure-strategy equilibrium bid for firms with the largest endowments of w is to always bid zero regardless of whether additional research is allowed by the sponsor following entry. But this obviously cannot be an equilibrium bidding strategy, so there is no efficient equilibrium for the uniform-price entry auction if firms differ in their initial starting positions. In a discriminatory auction, $\psi(w, w) = 0$, for all contestants holding endowments such that $H(w) \geq e^{-c/P}$, so there is no efficient equilibrium for the discriminatory auction either.

Proof of Theorem 5

If contestant i holds endowment w and bids as though holding $\hat{w}$, his expected profit from the tournament will be $\Pi(\hat{w}, w)$, as defined in equation (11) in the text. Checking first-order conditions and setting $\partial\Pi/\partial w|_{\hat{w}=w} = 0$, we get

$$B'(w) = \psi(w, w)h_{m:n-1}(w|w) + Kh_{m:n-1}(w|w).$$

Imposing $B(\underline{w}) = 0$, we arrive at equation (12) in the text. Since $\psi(w, w) \geq 0$ for all w, this bidding function is increasing in w. On the basis of the theorem by Guesnerie and Laffont (1984), this function is also a unique, pure-strategy bidding equilibrium whenever $\partial^2\Pi/\partial\hat{w}\partial w \geq 0$. This derivative is composed of two terms: one is positive and the other is ambiguous in sign. Independence sets the latter term to zero, so the overall sign is positive. Q.E.D.

Proof of Theorem 6

Integrate by parts to get

$$\int_{\underline{w}}^{\bar{w}} B(w)h(w)\,dw = \int_{\underline{w}}^{\bar{w}} [\psi(w, w) + K][1 - H(w)]h_{m:n-1}(w)\,dw.$$

When we substitute for order statistics, the expected bid is

$$\frac{m}{n}\int_{\underline{w}}^{\bar{w}} \psi(w, w)\,h_{m+1:n}(w)\,dw + \frac{mK}{n}.$$

The total expected revenues from all n bidders is

$$m\int_{\underline{w}}^{\bar{w}} \psi(w, w)\,h_{m+1:n}(w)\,dw + mK.$$

However, the sponsor must pay out mK to the m entrants, plus P to the overall winner. Therefore, the sponsor's total cost of conducting the tournament is

$$P - m\int_{\underline{w}}^{\bar{w}} \psi(w, w)\,h_{m+1:n}(w)\,dw,$$

which is independent of K and equivalent to the cost of using an efficient uniform-price auction. Q.E.D.

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