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A Theory of Bilateral Oligopoly, With Applications to Vertical Mergers

by

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Abstract: In horizontal mergers, concentration is often measured with the Hirschmann-Herfindahl Index (HHI). This index yields the price-cost margins in Cournot competition. In many modern merger cases, both buyers and sellers have market power, and indeed, the buyers and sellers may be the same set of firms. In such cases, the HHI is inapplicable. We develop an alternative theory that has similar data requirements as the HHI, applies to intermediate good industries with market power on both sides, and specializes to the HHI when buyers have no market power. The more inelastic is the downstream demand, the more captive production and consumption (not traded in the intermediate market) affects price/cost margins. The analysis is applied to the merger of the California gasoline refining and retail assets of Exxon and Mobil.

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The seven largest refiners of gasoline in California account for over 95% of the production of CARB (California Air Resources Board certified) gasoline sold in the state. The seven largest brands of gasoline also accounts for over 97% of retail sales of gasoline. Thus, the wholesale gasoline market in California is composed of large sellers and large buyers. What will be the effect of a merger of vertically integrated firms on the wholesale market? This question has immediate relevance with the combinations of Shell and Texaco, Exxon and Mobil, and BP/Amoco and Arco, all of which are completed or proposed.

Traditional antitrust analysis presumes dispersed buyers. Given such an environment, the Cournot model (quantity competition) suggests that the Hirschman-Herfindahl Index (HHI, which is the sum of the squared market shares of the firms) is proportional to the price-cost margin, which is the proportion of the price that is a markup over marginal cost. Specificially, the HHI divided by the elasticity of demand equals the price-cost margin. The HHI is zero for perfect competition and one for monopoly. The HHI has the major advantage of simplicity and low data requirements. In spite of well-publicized flaws, the HHI continues to be the workhorse of concentration analysis and is used by both the US Department of Justice and the Federal Trade Commission. The HHI is inapplicable, however, to markets where the buyers are concentrated.

When monopsony faces an oligopoly, as arises with U.S. Department of Defense purchases of high-technology weaponry, most analysts consider that the need for protecting the buyer from the exercise of market power is mitigated by the market power of the buyer. Thus, even when the buyers and sellers are separate firms, an analysis based on dispersed buyers is likely to err. How should antitrust authorities account for the market power of the Department of Defense in assessing the defense industry consolidation?

When both the buyers and sellers have the ability to influence price, one can consider merging a net buyer with a net seller, producing a more balanced firm, and bringing what was formerly traded in the intermediate good market inside the firm. Will this vertical integration reduce the exercise of market power and produce a more competitive market? Our objective in this paper is to offer an alternative to the HHI analysis with (i) similar informational requirements, (ii) the Cournot model as a special case, and (iii) an underlying game as plausible as the Cournot model.

The model we offer suffers from the same flaws as the Cournot model. It is highly stylized, and a static model. It uses a "black box" pricing mechanism motivated by the Cournot analysis. Our approach does not possess the Kreps and Scheinkman (1983) defense; however, that defense is not persuasive, as it depends on an unlikely and extreme rationing rule. Moreover, our model will suffer from the same flaws as the Cournot model in its application to antitrust analysis. Elasticities are treated as constants when they are not, and the relevant elasticities are taken as known.

In a traditional assessment of concentration according to the U.S. Department of Justice Merger guidelines, the firms' market shares are intended, where possible, to be shares of capacity. This is surprising in light of the fact that the Cournot model does *not* suggest the use of capacity shares in the HHI, but rather the share of sales in quantity units (not revenue). Like the Cournot model, the present study suggests using the sales data, rather than the capacity data, as the measure of market share. Capacity plays a role in our theory, and indeed a potential test of the theory is to check that actual capacities, where observed, are close to the capacities consistent with the theory.

The merger guidelines assess the effect of the merger by summing the market shares of the merging parties.[1] Such a procedure provides a useful approximation but is inconsistent with the theory (either Cournot or our theory), since the theory suggests that, if the merging parties' shares don't change, then the prices are unlikely to change as well. We advocate a more computationally-intensive approach, which involves estimating the capacities of the merging parties from the pre-merger market share data. Given those capacities, we then estimate the effect of the merger on the industry, taking into account the incentive of the merged firm to restrict output (or demand, in the case of buyers).

The Federal Communication Commission's sale of the PCS spectrum prompted a number of economists to espouse the view that it doesn't matter how the spectrum is sold; the resulting allocation will be efficient.[2] Underlying this view is an assumption of near-perfect competition. Given the slow development of nationwide roaming on the cellular spectrum, which was initially allocated by lottery, it seems empirically that the initial allocation matters, at least for a significant time. But how much? It is better to allocate the spectrum to big existing users, or to attempt to create a market in the intermediate good? Our theory permits assessment of market imperfections in intermediate goods such as spectrum.

Not surprisingly, there is a voluminous literature on the subject of vertical mergers. The models typically assign the market power either to buyers or to sellers, but not both.[3] These models are excellent for assessing some economic questions, including the incentive to raise rival's cost, the effects of contact in several markets, or the consequences of refusals-to-deal. In

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[1] Farrell and Shapiro (1990) and McAfee and Williams (1990) independently criticize the Cournot model while using a Cournot model to address the issue.
[2] See McMillan, 1994, Milgrom, 1998, and McAfee and McMillan, 1995 for a discussion of the auctions. The Fall, 1997 issue of the *Journal of Economics and Management Strategy* is devoted to the PCS auctions.
[3] See, for example, Hart and Tirole, 1990, Ordover, Saloner and Salop, 1990, Salinger, 1988, Salop and Scheffman, 1987, Bernheim and Whinston, 1990. An alternative to assigning the market power to one side of the market is Salinger's sequential model.

contrast, the present model will be more useful in practical applications. Klemperer and Meyer, 1989, provide a very general analysis of a closely related market game. The Klemperer and Meyer analysis motivates our concept of equilibrium; we have, however, substantially restricted the possible activities of the firms. These restrictions permit the calculation of antitrust effects in a practical way.

The next section presents a market game and derives price/cost margins and the value/price margin which is the equivalent for buyers. This section also derives the limiting results and the basic formulae. The third section analyzes the constant elasticity case. The fourth section investigates the effects of downstream market power on the analysis of section 2. The fifth section applies the analysis to the merger of the California assets of Exxon and Mobil, to illustrate the plausibility and applicability of the theory. The sixth section concludes.

2. The Theory

We begin with a standard model of buyers and sellers. There are n agents, indexed by i from 1 to n. Each agent i has a buying capacity k_i and a selling capacity γ_i. The agent will consume q_i and produce x_i. This produces utility

$$(1)\qquad \pi_i = k_i v\left({q_i}/{k_i} \right) - \gamma_i c\left({x_i}/{\gamma_i} \right) - p(q_i - x_i).$$

The parameter k_i indexes consumption capacity, and equation (1) reflects a general form of capacity, given that the value of consumption is homogeneous of degree one (constant returns to scale) in quantity q and capacity k. Similarly, γ_i is agent i's productive capacity. An agent with twice the productive capacity of another agent can produce twice as much at the same average cost. Such a formulation facilitates a consideration of mergers, for the merger of firms i and j produces a firm with consumption capacity $k_i + k_j$ and productive capacity $\gamma_i + \gamma_j$, and thereby is subject to the same analysis. Finally, equation (1) reflects pricing of the net purchase $q_i - x_i$, at a price that will be common among all the participants. It is assumed that v is concave and strictly increasing, while c is convex and strictly increasing.[4]

Assuming that all the agents have been identified, and there is no external source of production or consumption, supply and demand are equated with

$$(2) \qquad Q = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} x_i .$$

The efficient solution has the following, easily derived properties. Let

$$(3) \qquad K = \sum_{i=1}^{n} k_i, \quad \Gamma = \sum_{i=1}^{n} \gamma_i .$$

Then

$$(4) \qquad v'\left(Q/K\right) = c'\left(Q/\Gamma\right) = p,$$

and,

$$(5) \qquad q_i = \left(k_i/K\right)Q, \quad x_i = \left(\gamma_i/\Gamma\right)Q.$$

Define the standard market elasticities of demand and supply[5]:

$$(6) \qquad \varepsilon = -\left(\frac{P}{Q}\right)\frac{dQ^d}{dp} = \frac{-v'(Q/K)}{(Q/K)v''(Q/K)}, \quad \eta = \left(\frac{P}{Q}\right)\frac{dQ^s}{dp} = \frac{c'(Q/\Gamma)}{(Q/\Gamma)c''(Q/\Gamma)}.$$

Then a mechanical derivation shows

$$(7) \qquad \left(\frac{K}{Q}\right)\frac{\partial Q}{\partial K} = \frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}}, \left(\frac{\Gamma}{Q}\right)\frac{\partial Q}{\partial \Gamma} = \frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}}, and \left(\frac{K}{P}\right)\frac{\partial P}{\partial K} = -\left(\frac{\Gamma}{P}\right)\frac{\partial P}{\partial \Gamma} = \frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}.$$

Equations (2)-(7) provide the efficient, price-taking solution for agents with utility given by (1). Our interest, however, lies in the behavior of agents who are not price-takers, but

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[4] In addition, we assume $v'(q) \to 0$ and $c'(q) \to \infty$ as $q \to \infty$.

[5] The market demand function Q^d is given by $v'(Q^d/K) = p$. Supply is analogous.

recognize their ability to unilaterally influence the price. We model such agents by assuming that a market mechanism maps behavior into prices and quantities along the lines of the efficient solution. That is, each agent can behave as if they have capacities different from their actual capacities, and the market produces the efficient solution given that behavior.

In adopting this model, we do not permit agents to behave as if they have some other value function, other than the possible functions, which come in the form $kv(q/k)$. Similarly, the cost of production comes in the form $\gamma c(x/\gamma)$. In a mechanism design framework, agents can lie about their type, but they can't invent an impossible type. The admissible types in this model satisfy (1), and agents are assumed to be bound by (1). They can, however, act as if they have some type other than their true type.

This model can be viewed as turning the market mechanism into a black box, as in fact happens in the Cournot model, where the prices come from some un-modeled mechanism. (Kreps and Schienkman, 1983, provide an underlying model of Cournot pricing.) Given this "black box" approach, it seems appropriate to permit the market to be efficient when agents don't, in fact, exercise unilateral market power. Such considerations dictate setting a feasible set of agent types and having agents report a type to the market, which then dictates the competitive solution, given the types. Any other assumption would impose inefficiencies in the market mechanism, rather than having inefficiencies arise as the consequence of the rational exercise of market power by firms with a significant market presence. The only other restriction we have imposed is constant returns to scale in consumption and production.

The model can be interpreted as permitting agents to conceal demand (by acting as if they have a type $\hat{k}_i$ less than their true k_i) or exaggerate their costs (by acting as if their capacity γ_i is less than it in fact is). We will, however, make their actual types common knowledge in the solution. Thus, the formal solution involves agents who make reports of types to a mechanism, where a type is a pair $(\hat{k}_i, \hat{\gamma}_i)$, and then the mechanism takes these reports and maps to a market outcome, which is given by (2)-(7). In choosing their reports, agents are assumed to know the true types of other agents. As with any equilibrium notion under complete information, agents correctly guess the reports of other agents. We use the Nash equilibrium concept to characterize agents' reports.

In order to characterize the solution to our market game, it is helpful to introduce a bit of notation. A hat over a variable indicates that it is a report, rather than a true value. Capital letters are used to denote market aggregates (K, Γ, Q); hats are dispensed with because such variables will always be derived from reports. The market share of firm i in consumption is denoted s_i and in production σ_i:

(8) $$s_i = \hat{k}_i / K = \hat{k}_i \Big/ \sum_{j=1}^{n} \hat{k}_j , \qquad \sigma_i = \hat{\gamma}_i / \Gamma = \hat{\gamma}_i \Big/ \sum_{j=1}^{n} \hat{\gamma}_j .$$

Agent i obtains, given reports $(\hat{k}_i, \hat{\gamma}_i)$:

(9) $$\pi_i = k_i\, v\left(\frac{\hat{k}_i Q / K}{k_i} \right) - \gamma_i\, c\left(\frac{\hat{\gamma}_i Q / \Gamma}{\gamma_i} \right) - p\left(\frac{\hat{k}_i}{K} Q - \frac{\hat{\gamma}_i}{\Gamma} Q\right)$$

$$= k_i\, v\left(\frac{s_i Q}{k_i} \right) - \gamma_i\, c\left(\frac{\sigma_i Q}{\gamma_i} \right) - pQ(s_i - \sigma_i).$$

Let

(10) $$v_i' = v'\left(s_i Q / k_i \right) \textit{and } c_i' = c'\left(\sigma_i Q / \gamma_i \right)$$

Equations (10) give the realized marginal value and marginal cost. In the efficient solution, these of course both equal the price.

Theorem 1: *In any interior equilibrium,* $v_i' = c_i'$ *and*

$$(11) \qquad \frac{v_i' - p}{p} = \frac{c_i' - p}{p} = \frac{s_i - \sigma_i}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}.$$

All proofs are contained in the appendix.

There are two immediate observations. First, each agent is technically efficient about its production; that is, its marginal cost equals the agent's marginal value. Thus, the agent can't, in the equilibrium allocation, gain from secretly producing more and consuming that output. This is *not* to say that the agent couldn't gain from the ability to secretly produce and consume, for the agent might gain from doing so when altering their reports appropriately. Second, net buyers value the good more than the price, and net sellers value the good less than price. Thus, net buyers restrict their demand below that which would arise in perfect competition, and net sellers restrict their supply. In both cases, the gain arises because of price effects.

We show in the appendix that (11) generalizes, when one of the choices is not in the interior, in the usual complementary-slackness way. For example, if s_i=0, (11) becomes

$$(11') \qquad \frac{v'(0) - p}{p} \le \frac{c_i' - p}{p} = \frac{-\sigma_i}{\varepsilon + \eta(1 - \sigma_i)}.$$

This is the natural generalization of (11).

In the Cournot model, any merger caused prices to rise. Firms with larger market shares distort away from efficiency by a larger amount. In the present model, in contrast, it is firms whose production and consumption aren't balanced who distort the most. Moreover, a merger of a balanced firm (with zero distortion) and an unbalanced firm may increase the distortion of the unbalanced firm, by an increase in the denominator of equation (11).

Theorem 2: *Suppose sellers are labelled* 1 *to m and have no consumption* (k_i=0), *buyers are labelled m to n and have no production* (γ_i=0). *Then, in the limit as the buyers get small holding constant K, buyers have* $v_i' = p$ *and*

$$\text{(12)} \qquad \frac{p - c_i'}{p} = \frac{\sigma_i}{\varepsilon + \eta(1 - \sigma_i)}.$$

When η=0, equation (12) is precisely the Lerner Index or price/cost margin equation for the Cournot model. The reason for the difference between the present model and the Cournot model is that a seller who restricts supply expects a portion of that supply to be made up by other sellers, since other sellers are represented by increasing supply curves rather than constant quantities. As $c''/c' \to \infty, \eta \to 0$, and the Cournot outcome arises.

Theorem 3: *The (quantity weighted) average difference between marginal valuations and marginal costs satisfies:*

$$\text{(13)} \qquad \frac{1}{p}\left(\sum_{i=1}^{n} s_i v_i' - \sum_{i=1}^{n} \sigma_i c_i'\right) = \sum_{i=1}^{n} \left(\frac{(s_i - \sigma_i)^2}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}\right)$$

Equation (13) is the equivalent of the Hirschman-Herfindahl Index for the present model. It has the same useful features -- it depends only on market shares and elasticities. Unless the elasticities are equal, it depends on both production and consumption. As noted above, zero net demand causes no inefficiency. However, with even a small but nonzero net demand or supply, size exacerbates the inefficiency.

In this framework, the shares are of production or consumption, and not capacity. The U.S. Department of Justice Merger Guidelines (1992) generally calls for evaluation shares of capacity. While our analysis begins with capacities, the shares are actual shares of production (σ_i) or consumption (s_i), rather than the capacity for production and consumption, respectively. The use of actual consumption and production is an advantageous feature of the theory, since these values tend to be readily observed, while capacities are not. Moreover, capacity is often subject to vociferous debate by economic analysts, while the market shares may be more readily observable. Finally, the shares are shares of the total quantity and not revenue shares. However, like the Cournot model, our model is not designed to handle industries with differentiated products, which is the situation where a debate about revenue versus quantity shares arises.

3. The Constant Elasticity Case

The case where the demand and supply elasticities are constant is especially informative. In particular, even when elasticities vary, formulae derived from the constant elasticity case apply approximately, with the error determined by the amount of variation in the elasticities. With constant elasticities,

$$v'(q) = q^{-e^{-1}},\ c'(q) = q^{h^{-1}}. \tag{14}$$

Let Q_f represent the first best quantity, that which arises when all firms are sincere in their behavior, and p_f be the associated price. Then

Theorem 4: *With constant elasticities, the size of the firms' misrepresentations is given by*

$$\frac{\hat{k}_i}{k_i} = \left(1 + \frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i) + \boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{-e},\ \frac{\hat{g}_i}{g_i} = \left(1 + \frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i) + \boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{h}. \tag{15}$$

Moreover,

$$\frac{Q_f}{Q} = \left[\sum_{i=1}^{n} s_i \left(1 + \frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i) + \boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{e}\right]^{\frac{h}{e+h}} \left[\sum_{i=1}^{n} \boldsymbol{s}_i \left(1 + \frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i) + \boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{-h}\right]^{\frac{e}{e+h}}, \tag{16}$$

Finally,

$$(17) \qquad \frac{p}{p_f} = \left(\frac{\sum_{i=1}^{n} \sigma_i \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{-\eta}}{\sum_{i=1}^{n} s_i \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{\varepsilon}} \right)^{\frac{1}{\varepsilon+\eta}} .$$

Equation (15) confirms the intuition that the misrepresentation is largest for the largest net traders, and small for those not participating significantly in the intermediate good market. Indeed, the size of the misrepresentation is proportional to the discrepancy between price and marginal value or cost, as given by (12), adjusted for the demand elasticity. This is hardly surprising, since the constant demand and supply elasticities insure that marginal values can be converted to misrepresentations in a log-linear fashion.

Equation (16) provides the formula for lost trades. Here there are two effects. Net buyers under-represent their demand, but over-represent their supply. On balance, net buyers under-represent their net demands, which is why the quantity-weighted average marginal value exceeds the quantity-weighted average marginal cost. Equation (16) provides a straightforward means of calculating the extent to which a market is functioning inefficiently, both before and after a merger, at least in the case where the elasticities are approximately constant.

Equation (17) gives the effect of strategic behavior in the model on price. Note that the price can be larger, or smaller, than the efficient full-information price. Market power on the buyer's side (high values of s_i) tend to decrease the price, with buyers exercising market power. Similarly, as σ_i increases, the price tends to rise. In this model, an anticompetitive effect of a merger is felt through quantity rather than price. If the seller's side is concentrated and two net buyers merge, the effect may be to bring the price closer to the efficient price, but with a reduction in quantity.

4. Downstream Concentration

In many, perhaps even most, applications, the assumption that a buyer in the intermediate good market can safely ignore the behavior of other firms in calculating the value of consumption is unfounded. In this section, we presume that the payoff to firms is Cournot-like, that is, the value of consumption is given by

(18) $$v_i = r(Q)q_i - k_i w\left(q_i / k_i \right)$$

Here, r gives the downstream, final good price, and w accounts for a selling cost. This model reverts to the previous analysis when r is constant. In general, the model accommodates downstream effects via the demand function r. Firm profits are:

(19) $$\pi_i = r(Q)q_i - k_i w\left(q_i / k_i \right) - \gamma_i c\left(x_i / \gamma_i \right) - p(q_i - x_i).$$

As before, we calculate the efficient solution, which satisfies:

(20) $$p = c'(Q/\Gamma) = c'(q_i/\gamma_i)$$

and

(21) $$r(Q) = p + w'(Q/K) = p + w'(q_i/k_i).$$

Let α be the elasticity of demand, and β be the elasticity of the selling cost w. Let θ be ratio of the intermediate good price p to the final good price r. The observables of the analysis will be the market shares (both production, σ_i, and retail, s_i), the elasticity of final good demand, α, of selling cost, β, of production cost, η and the price ratio $\theta = p/r$. It will turn out that the elasticities enter in a particular way, and thus it is useful to define:

(22) $$A = \alpha^{-1};\ B = (1-\theta)\beta^{-1};\ C = \theta\eta^{-1}.$$

We replicate the analysis of section 2 in the appendix for this more general model. The structure is to use equations (20) and (21) to construct the value to each firm of reports of k_i and γ_i. The first order conditions provide necessary conditions for a Nash equilibrium to the reporting game. These first order conditions are used to compute the price/cost margin, weighted by the firm shares. In particular, we look for a modified herfindahl index (MHI) given by:

(23) $$MHI = \sum_{i=1}^{n} \frac{1}{r}\left((r(Q) - p - w_i')s_i + (p - c_i')\sigma_i \right).$$

In (23), the notation $w_i' = w'\left(q_i / \hat{k}_i\right)$, $c_i' = c'\left(x_i / \hat{\gamma}_i\right)$ is used as before.

The main theorem characterizes the modified Herfindahl index for an interior solution.

Theorem 5: *In an interior equilibrium,*

(24) $$MHI = \sum_{i=1}^{n} \left[\frac{BC(s_i - \sigma_i)^2 + ABs_i^2(1-\sigma_i) + AC\sigma_i^2(1-s_i)}{A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)} \right].$$

While complex in general, this formula has several important special cases. If A=0, the downstream market has perfectly elastic demand. As a result, $k_i v(q_i/k_i) = rq_i - k_i w(q_i/k_i)$ provides the connection between the previous model and the present model, and (24) readily reduces to (13).

When B=0, there is a constant retailing cost w. The case of B=0 is analogous to Cournot, in that all firms are equally efficient at selling, although the firms vary in their efficiency at producing. In this case, (24) reduces to

(25) $$MHI\big|_{B=0} = \sum_{i=1}^{n} \left[\frac{AC\sigma_i^2}{A(1-\sigma_i) + C} \right] = \sum_{i=1}^{n} \left[\frac{\theta\sigma_i^2}{\eta(1-\sigma_i) + \theta\alpha} \right]$$

The Herfindahl index reflects the effect of the wholesale market through the elasticity of supply η. If η=0, the Cournot HHI arises. For positive η, the possibility of resale increases the price/cost margin. This arises because a firm with a large capacity now has an alternative to selling that capacity on the market. A firm with a large capacity can sell some of its Cournot level of capacity to firms with a smaller capacity. The advantage of such sales to the large firm is the reduction in desire of the smaller firms to produce more, which helps increase the retail price. In essence, the larger firms buy off the smaller firms via sales in the intermediate good market, thereby reducing the incentive of the smaller firms to increase their production.

The formula (24) can be decomposed into Herfindahl-type indicies for three separate markets: transactions, production and consumption. Note

$$(26) \qquad MHI = \sum_{i=1}^{n} \left[\frac{B(1-\sigma_i)+C(1-s_i)}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \quad \frac{(s_i-\sigma_i)^2}{\eta(1-\sigma_i)/\theta+\beta(1-s_i)/(1-\theta)} \right].$$

$$+\sum_{i=1}^{n} \left[\frac{A(1-s_i)(1-\sigma_i)}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \left((1-\theta)\frac{s_i^2}{\beta(1-s_i)} + \theta \frac{\sigma_i^2}{\eta(1-\sigma_i)} \right) \right].$$

The modified herfindahl index, MHI, is an average of three separate indicies. The first index corresponds to the transactions in the intermediate good market. In form, this term looks like the expression in Theorem 3, adjusted to express the elasticities in terms of the final output prices. The second expression is an average of the indicies associated with production and consumption of the intermediate good. These two indicies ignore the fact that firms consume some of their own production.[6]

When the downstream market is very elastic, A is near zero. In this case, the MHI reduces to that of Theorem 3. This occurs because elastic demand in the downstream market eliminates downstream effects, so that the only effects arise in the intermediate good market. In contrast, when the downstream market is relatively inelastic, downstream effects dominate, and

[6] The MHI is, on a term by term basis, a weighted average of these three indicies. However, the weights in formula (26) vary with *i*, except in limiting cases.

the MHI is approximately an average of the herfindahl indicies for the upstream and downstream markets, viewed as separate markets.

In some sense, these limiting cases provide a resolution of the question of how to treat captive consumption. When demand is very inelastic, as with gasoline in California, then the issue of captive consumption can be ignored without major loss: it is gross production and consumption that matter. In this case, it is appropriate to view the upstream and downstream markets as separate markets and ignore the fact that the same firms may be involved in both. In particular, a merger of a pure producer and a pure retailer should raise minimal concerns. On the other hand, when demand is very elastic (A near zero), gross consumption and gross production can be safely ignored, and the market treated as if the producers and consumers of the intermediate good were separate firms, with net trades in the intermediate good the only issue that arises.[7] Few real world cases are likely to approximate the description of very elastic market demand.[8] However, the case of A=0 also corresponds to the case where the buyers do not compete in a downstream market, and thus may have alternative applications.

In the appendix, we provide the formulae governing the special case of constant elasticities. It is straightforward to compute the reduction in quantity that arises from a concentrated market, as a proportion of the fully efficient, first-best quantity. Moreover, we provide a Mathematica 3.0 program which takes market shares as inputs and computes the capital shares of the firms, the quantity reduction and the effects of a merger.[9]

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[7] However, the denominator still depends on gross production and consumption, rather than net production and consumption. This can matter when mergers dramatically change market shares, and even the merger of a pure producer and pure consumer can have an effect.

[8] When market demand is very elastic, it is likely that there are substitutes that have been ignored. It would usually be preferable to account for such substitutes in the market, rather than ignore them.

[9] This program is available on McAfee's website, http://www.eco.utexas.edu/faculty/mcafee/.

5. Merger Analysis

In this section, we apply the analysis to a merger of Exxon and Mobil's gasoline refining and retailing assets in California. California's gasoline market is relatively isolated from the rest of the nation, both because of transportation costs,[10] and because of the California Air Resources Board's requirement of gasoline reformulated for lower emission, a type of gasoline known as CARB.

Table 1: Approximate Market Shares, California CARB Gasoline
Post-Merger Numbers in Parentheses

Company	i	Refining Market Share (σ_i)	Refining Capital Share	Retail Market Share (s_i)	Retail Capital Share
Chevron	1	26.4 (26.6)	29.5 (29.5)	19.2 (19.5)	19.0 (19.0)
Tosco	2	21.5 (21.7)	21.7 (21.7)	17.8 (18.0)	17.8 (17.8)
Equilon	3	16.6 (16.7)	16.1 (16.1)	16.0 (16.2)	16.0 (16.0)
Arco	4	13.8 (13.9)	13.0 (13.0)	20.4 (20.7)	22.0 (22.0)
Mobil	5	7.0 (13.3)	6.2 (12.4)	9.7 (17.5)	9.3 (17.8)
Exxon	6	7.0 (0.0)	6.2 (0.0)	8.9 (0.0)	8.5 (0.0)
Ultramar	7	5.4 (5.4)	4.7 (4.7)	6.8 (6.9)	6.4 (6.4)
Paramount	8	2.3 (2.3)	2.0 (2.0)	0.0 (0.0)	0.0 (0.0)
Kern	9	0.0 (0.0)	0.0 (0.0)	0.3 (0.3)	0.27 (0.27)
Koch	10	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
Vitol	11	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
Tesoro	12	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
PetroDiamond	13	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)
Time	14	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)
Glencoe	15	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)

Available market share data is generally imperfect, because of variations due to shut-downs and measurement error, and the present analysis should be viewed as an illustration of the theory rather than a formal analysis of the Exxon-Mobil merger. Nevertheless, we have tried to use the best available data for the analysis. In Table 1, we provide a list of market shares, along

[10] There is currently no pipeline permitting transfer of Texas or Louisiana refined gasoline to California, and the Panama Canal can not handle large tankers, and in any case is expensive. Nevertheless, when prices are high enough, CARB gasoline has been brought from the Hess refinery in the Caribbean.

with our estimates of the underlying capital shares and the post-merger market shares, which will be discussed below. The data come from Leffler and Pulliam, 1999.

From Table 1, it is clear that there is a significant market in the intermediate good of bulk (unbranded) gasoline, prior to branding and the addition of proprietary additives. However, the actual size of the intermediate good market is larger than one might conclude from Table 1, because firms engage in swaps. Swaps trade gasoline in one region for gasoline in another. Since swaps are balanced, they will not affect the numbers in Table 1.

It is well known that the demand for gasoline is very inelastic. We consider a base case of an elasticity of demand, α, of 1/3. We estimate θ to be 0.7, an estimate derived from an average of 60.1 cents spot price for refined CARB gasoline, out of an average of 85.5 (net of taxes) at the pump.[11] We believe the selling cost to be fairly elastic, with a best estimate of β=5. Similarly, by all accounts refining costs are quite inelastic; we use η=½ as the base case. We will consider the robustness to parameters below, with α=1/5, β=3, and η=1/3.

Table 2: Markups and Quantity Reduction, in Percent, for a Symmetric Industry

Number of Firms	2	3	4	5	6	7	8	9	10	15	20
Average Markup	74.0	42.3	30.0	22.7	18.4	15.5	13.4	11.8	10.5	6.8	5.1
Q/Q_f, in percent	82.9	87.7	92.2	94.2	95.4	96.2	96.8	97.2	97.5	98.4	98.8

Prior to analyzing the effect of a merger of Exxon and Mobil, we present the markup in a fully symmetric industry, given our base case assumptions. A symmetric industry involves no trade in the intermediate good. We consider variations in the number of identical firms, as a way of benchmarking the price-cost margins. These findings are presented in Table 2. Thus, a price-cost margin of 11.8 percent roughly corresponds to a symmetric industry with 9 firms, dropping

[11] We will use all prices net of taxes. As a consequence, the elasticity of demand builds in the effect of taxes, so that a 10% retail price increase (before taxes) corresponds approximately to an 17% increase in the after tax price. Thus, the elasticity of 1/3 corresponds to an actual elasticity of closer to 0.2.

to 6.8% with 15 identical firms. We generally find the gasoline industry in California to have 20% margins and approximately 95% efficiency, which is similar to an industry with 6 firms.

Table 2 also presents the calculations, for the base case elasticities, of the reduction in output for a symmetric industry. A symmetric duopoly would reduce quantity by 17.1%, which would create price increases around 75% over the competitive level, given the demand elasticity of 1/3. Even with fifteen identical firms, the quantity is reduced by 3.6%, which creates an approximately 10.8% increase in the retail price over the efficient quantity.

Table 3 presents our summary of the Exxon/Mobil merger. The first three columns provide the assumptions on elasticities that define the four rows of calculations. The fourth column provides the markups that would prevail under a fully symmetric and balanced industry, that is, one comprised of fifteen equal sized firms. This is the best outcome that can arise in the model, given the constraint of fifteen firms, and can be used a benchmark. The fifth column considers a world without refined gasoline exchange, in which all fifteen companies are balanced, and is created by averaging production and consumption shares for each firm. This calculation provides an alternative benchmark for comparison, to assess the inefficiency of the intermediate good exchange. The next four columns use the existing market shares, reported in Table 1, as an input, and then compute the price-cost margin and quantity reduction, pre-merger, post-merger, with a refinery sale, or with a sale of retail outlets, respectively.

Table 3 does not use the naive approach of combining Exxon and Mobil's market shares, an approach employed in the Department of Justice Merger Guidelines. In contrast to the merger guidelines approach, we first estimate the capital held by the firms, then combine this capital in the merger, then compute the equilibrium given the post-merger allocation of capital. The estimates are not dramatically different than those that arise using the naive approach of the merger guidelines. To model the divestiture of refining capacity, we combine only the retailing capital of Exxon and Mobil; similarly, to model the sale of retailing, we combine the refining capacity.

The estimated shares of capital are presented in Table 1. These capital shares reflect the incentives of large net sellers in the intermediate market to reduce their sales in order to increase the price, and the incentive of large net buyers to decrease their demand to reduce the price. Equilon, the firm resulting from the Shell-Texaco merger, is almost exactly balanced and thus its capital shares are relatively close to its market shares. In contrast, a net seller in the intermediate market like Chevron refines significantly less than its capital share, but retails close to its retail capital share. Arco, a net buyer of unbranded gasoline, sells less than its share from its retail stores, but refines more to its share of refinery capacity. The estimates also reflect the incentives of all parties to reduce their downstream sales to increase the price, an incentive that is larger the larger is the retailer.

Table 3: Analysis of Exxon – Mobil Merger

Cases			Markup as a Percent of Retail Price (Quantity as a Percent of Fully Efficient Quantity in Parentheses)					
α	β	η	Symmetric	Balanced Asymmetric	Pre-merger Markup	Post-merger Markup	Refinery Sale	Retail Sale
1/3	5	½	6.9 (98.4)	18.4 (95.3)	20.0 (94.6)	21.3 (94.3)	20.1 (94.6)	21.2 (94.3)
1/5	5	½	7.9 (98.7)	21.6 (96.0)	23.6 (95.4)	25.2 (95.2)	23.7 (95.4)	25.2 (95.2)
1/3	3	½	7.0 (98.4)	18.7 (95.3)	20.3 (94.6)	21.7 (94.3)	20.5 (94.6)	21.6 (94.4)
1/3	5	1/3	8.7 (98.2)	23.0 (94.6)	25.1 (93.8)	26.7 (93.5)	25.2 (93.8)	26.7 (93.5)

The sixth column of Table 3 provides the pre-merger markup, or MHI, and is a direct calculation from equation (24) using the market shares of Table 1. The seventh, eighth and ninth columns combine Exxon and Mobil's capital assets in various ways. The seventh combines both retail and refining capital. The eighth combines the retail capital, but leaves Exxon's Benicia refinery in the hands of an alternative supplier not listed in the table. This corresponds to a sale of the Exxon refinery. The ninth and last column considers the alternative of a sale of Exxon's retail outlets. (It has been announced that Exxon will sell both its refining and retailing operations in California.)

Our analysis suggests that without divestiture the merger will, under the hypotheses of the theory, have a small effect on the retail price. In the base case, the markup increases from 20% to 21%, and the retail price increases 1%.[12] Moreover, a sale of a refinery eliminates most the price increase; the predicted price increase is less than a mil. Unless retailing costs are much less elastic than we believe, a sale of retail outlets accomplishes very little. The predicted changes in prices, as a percent of the pre-tax retail price, are summarized in Table 4.

Table 4: Analysis of Exxon – Mobil Merger

Cases			Expected Percentage Quantity Decrease (Percentage Price Increase in Parentheses)		
α	β	η	Full Merger	Refinery Sale	Retail Sale
1/3	5	½	0.31 (0.94)	0.03 (0.09)	0.30 (0.90)
1/5	5	½	0.27 (1.36)	0.02 (0.11)	0.25 (1.29)
1/3	3	½	0.32 (0.97)	0.05 (0.15)	0.30 (0.89)
1/3	5	1/3	0.35 (1.06)	0.03 (0.08)	0.34 (1.03)

The predicted quantity, as a percentage of the fully efficient quantity, is presented in Table 3, in parentheses. The first three columns present the prevailing parameters. The next six columns correspond to the conceptual experiments discussed above. The symmetric column considers fifteen equal sized firms. The balanced asymmetric column uses the data of Table 1, but averages the refining and retail market shares to yield a no-trade initial solution. The pre-merger column corresponds to Table 1; post-merger combines Exxon and Mobil. Finally, the last two columns consider a divestiture of a refinery and retail assets, respectively. We see the

[12] The percentage increase in the retail price can be computed by noting that $p=q^{-A}$.

effects of the merger through a small quantity reduction. Again, we see that a refinery sale eliminates nearly all of the quantity reduction.

The analysis used the computed market shares rather than the approach espoused by the U.S. Department of Justice Merger Guidelines. Our approach is completely consistent with the theory, unlike the merger guidelines approach, which sets the post-merger share of the merging firms to the sum of their pre-merger shares. This is inconsistent with the theory because the merger will have an impact on all firms' shares. In Table 1, we provide our estimate of the post-merger shares along side the pre-merger shares. Exxon and Mobil were responsible for 18.6% of the refining, and we estimate that the merger will cause them to contract to 17.4%. The other firms increase their share, though not enough to offset the combined firm's contraction.

There is little to be gained by using the naive merger guidelines market shares, because the analysis is sufficiently complicated to require machine-based computation. (Such programs are simple, however, and one is provided in the Appendix.) However, we replicated the analysis using the naive market shares, and the outcomes are virtually identical. Thus, it appears that the naive approach gives the right answer in this application.

7. Conclusion

This paper presents a method for measuring industry concentration in intermediate goods markets. It is especially relevant when firms have captive consumption, that is, some of the producers of the intermediate good use some or all of their own production for downstream sales.

The major advantages of the theory are its applicability to a wide variety of industry structures, its low informational requirements, and its relatively simple formulae. The major disadvantages are the special structure assumed in the theory, and the static nature of the analysis. The special structure mirrors Cournot, and thus is subject to the same criticisms as the

Cournot model. For all its defects, the Cournot model remains the standard model for antitrust analysis; the present theory extends Cournot-type analysis to a new realm.

We considered the application of the theory to the California assets of Exxon and Mobil. Several reasonable predictions emerge. First, the industry produces around 95% of the efficient quantity and the merger reduces quantity by a small amount, around 0.3%. Second, the price-cost margin is on the order of 20% and rises by a percentage point or two. Third, a sale of Exxon's refinery eliminates nearly all of the predicted price increase. This last prediction arises because retailing costs are relatively elastic, so that firms are fairly competitive downstream. Thus, the effects of industry concentration arise primarily from refining, rather than retail. Hence the sale of a refinery (Exxon and Mobil have one refinery each) cures most of the problem associated with the merger. Fourth, the naive approach based purely on market shares gives answers similar to the more sophisticated approach of first computing the capital levels, combining the capital of the merging parties, then computing the new equilibrium market shares. Finally, it is worth noting that the computations associated with the present analysis are straightforward, and run in a few seconds on a modern PC.

As with Cournot analysis, the static nature of the theory is problematic. In some industries, entry of new capacity is sufficiently easy that entry would undercut any exercise of market power. The present theory does not accommodate entry, and thus any analysis would need a separate consideration of the feasibility and likelihood of timely entry.[13] When entry is an important consideration, the present analysis provides an upper bound on the ill-effects of merger.

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[13] McAfee, Simons and Williams, 1992, present a Cournot-based merger evaluation theory that explicitly accommodates entry in the analysis.

References

Bernheim, Douglas, and Michael Whinston, "Multimarket Contact and Collusive Behavior," *Rand Journal of Economics*, 21, no. 1, Spring, 1990, pp. 1- 26.

Farrell, Joseph, and Carl Shapiro, "Horizontal Mergers: An Equilibrium Analysis," *American Economic Review,* 80, p.107-26.

Hart, Oliver and Jean Tirole, "Vertical Integration and Market Foreclosure," *Brookings Papers on Economic Activity: Microeconomics,* Special Issue, 1990, 205-76.

Klemperer, Paul and Margaret Meyer, "Supply Function Equilibria in Oligopoly Under Uncertainty," *Econometrica* 57, no. 6, November 1989, 1243-77.

Kreps, David and Jose Scheinkman, "Quantity Precommitment and Betrand Competition Yield Cournot Outcomes," *Bell Journal of Economics*, 14, no. 2, Autumn 1983, 326-37.

Leffler, Keith and Barry Pulliam, “Preliminary Report to the California Attorney General Regarding California Gasoline Prices,” November 22, 1999.

R. Preston McAfee and John McMillan, "Analyzing the Airwaves Auction," *Journal of Economic Perspectives,* 10, no. 1, Winter 1996, 159-75.

McAfee, R. Preston and Michael Williams, "Horizontal Mergers and Antitrust Policy," *Journal of Industrial Economics,* XL, June, 1992, 181-7.

McAfee, R. Preston, Joseph Simons and Michael Williams, “Horizontal Mergers in Spatially Differentiated Noncooperative Markets,” *Journal of Industrial Economics XL*, December 1992, 349-57.

McMillan, John, "Selling Spectrum Rights," *Journal of Economic Perspectives,* 8, Summer 1994, 145-62.

Milgrom, Paul, "Game Theory and the Spectrum Auctions," *European Economic Review*, 42, May, 1998, pp. 771- 78.

Ordover, Janusz, Garth Saloner and Steven Salop, "Equilibrium Vertical Foreclosure," *American Economic Review,* 80, no. 1, March 1990, p. 127-42.

Salinger, Michael, "Vertical Mergers and Market Foreclosure," *Quarterly Journal of Economics,* 103, no.2, May 1988, 345-56.

Salop, Steven, and David Scheffman, "Cost Raising Strategies," *Journal of Industrial Economics,* 36, no.1, September 1987, 19-34.

U.S. Department of Justice Merger Guidelines.

Appendix

Proof of Theorem 1:

$$\frac{\partial \pi_i}{\partial \hat{k}_i} = \left(v'\!\left(\frac{s_i Q}{k_i}\right) - p \right)\left(Q\left(\frac{1}{K} - \frac{\hat{k}_i}{K^2}\right) + s_i \frac{\partial Q}{\partial K} \right) + \left(p - c'\!\left(\frac{\sigma_i Q}{\gamma_i}\right)\right)\left(\sigma_i \frac{\partial Q}{\partial K}\right) - Q(s_i - \sigma_i)\frac{\partial p}{\partial K}$$

$$= \frac{Q}{K}\left[(v_i' - p)\left((1 - s_i) + s_i \frac{\varepsilon^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right) + (p - c_i')\left(\sigma_i \frac{\varepsilon^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right) - p(s_i - \sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right].$$

Similarly,

$$\frac{\partial \pi_i}{\partial \hat{\gamma}_i} = \left(v'\!\left(\frac{s_i Q}{k_i}\right) - p \right)\left(s_i \frac{\partial Q}{\partial \Gamma} \right) + \left(p - c'\!\left(\frac{\sigma_i Q}{\gamma_i}\right)\right)\left(Q\left(\frac{1}{\Gamma} - \frac{\hat{\gamma}_i}{\Gamma^2}\right) + \sigma_i \frac{\partial Q}{\partial \Gamma}\right) - Q(s_i - \sigma_i)\frac{\partial p}{\partial \Gamma}$$

$$= \frac{Q}{\Gamma}\left[(v_i' - p)\left(s_i \frac{\eta^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right) + (p - c_i')\left(1 - \sigma_i + \sigma_i \frac{\eta^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right) + p(s_i - \sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right].$$

Thus,

$$\frac{K}{Q}\frac{\partial \pi_i}{\partial \hat{k}_i} + \frac{\Gamma}{Q}\frac{\partial \pi_i}{\partial \hat{\gamma}_i} = v_i' - c_i'.$$

In an interior equilibrium, then, $v_i' - c_i' = 0$. With either of the first order conditions, we obtain (11). Q.E.D.

The Case of $s_i=0$, $\sigma_i>0$.

If $\sigma_i>0$, the first order condition on $\hat{\gamma}_i$ holds with equality. Consequently, using $s_i=0$,

$$0 = \frac{Q}{\Gamma}\left[(p - c_i')\left(1 - \sigma_i + \sigma_i \frac{\eta^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right) + p(0 - \sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1} + \eta^{-1}} \right]$$

This yields

$$\frac{p - c_i'}{p} = \frac{\sigma_i}{\varepsilon + \eta(1 - \sigma_i)},$$

a formula that respects (11) (for $s_i=0$). In addition, we have

$$0 \geq \left.\frac{\partial \boldsymbol{p}_i}{\partial \hat{k}_i}\right|_{s_i=0},$$

which gives

$$\frac{v'(0)-p}{p} \leq \frac{-\boldsymbol{s}_i}{\boldsymbol{e}+\boldsymbol{h}(1-\boldsymbol{s}_i)}.$$

Summarizing,

$$\frac{v_i'-p}{p} \leq \frac{s_i-\boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)},\ \textit{with equality if}\ \ s_i > 0.$$

and

$$\frac{p-c_i'}{p} \leq \frac{s_i-\boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)},\ \textit{with equality if}\ \ \boldsymbol{s}_i > 0.$$ Q.E.D.

Proof of Theorem 2:

Suppose k_i=0. Then $kv(q/k) = \dfrac{v(q/k)}{1/k} \approx qv'(q/k) \xrightarrow[k\to 0]{} 0$. Thus, an agent with k_i=0 will report $\hat{k}_i = 0$. Similarly, an agent with γ_i=0 produces zero. This yields (12). As buyers get small, holding constant K, (11) gives $v_i' = p$ for the buyers. Q.E.D.

Proof of Theorem 3: It is readily checked that the following substitutions hold, even when a share is zero.

$$\sum_{i=1}^n v_i' s_i - \sum_{i=1}^n c_i' \boldsymbol{s}_i = \sum_{i=1}^n (v_i'-p)s_i + \sum_{i=1}^n (p-c_i')\boldsymbol{s}_i$$

$$= \sum_{i=1}^n (v_i'-p)s_i + \sum_{i=1}^n (p-v_i')\boldsymbol{s}_i = \sum_{i=1}^n (v_i'-p)s_i - \sum_{i=1}^n (v_i'-p)\boldsymbol{s}_i$$

$$= \sum_{i=1}^n (v_i'-p)(s_i-\boldsymbol{s}_i) = \sum_{i=1}^n \frac{p(s_i-\boldsymbol{s}_i)^2}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}.$$ Q.E.D.

Proof of Theorem 4: Applying (4), (11), (14):

$$\left(\frac{s_i Q}{k_i}\right)^{-\frac{1}{e}} = v'(\frac{s_i Q}{k_i}) = p\left(1+\frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right) = v'\left(\frac{Q}{\sum_{i=1}^{n}\hat{k}_i}\right)\left(1+\frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)$$

This readily gives the first part of (15); the second half is symmetric.

Rewrite (15) to obtain

$$k_i = \hat{k}_i\left(1+\frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{e}.$$

Thus

$$\frac{k_i}{\sum_{j=1}^{n}\hat{k}_j} = s_i\left(1+\frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{e}.$$

Thus

$$\frac{\sum_{i=1}^{n}k_i}{\sum_{i=1}^{n}\hat{k}_i} = \sum_{i=1}^{n}s_i\left(1+\frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{e}.$$

A similar calculation gives

$$\frac{\sum_{i=1}^{n}g_i}{\sum_{i=1}^{n}\hat{g}_i} = \sum_{i=1}^{n}\boldsymbol{s}_i\left(1+\frac{s_i - \boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{-h}.$$

Note that, with constant elasticity, actual quantity is

$$Q = \left(\sum_{i=1}^{n}\hat{k}_i\right)^{\frac{h}{e+h}}\left(\sum_{i=1}^{n}\hat{g}_i\right)^{\frac{e}{e+h}},$$

and

$$Q_f = \left(\sum_{i=1}^{n}k_i\right)^{\frac{h}{e+h}}\left(\sum_{i=1}^{n}g_i\right)^{\frac{e}{e+h}}.$$

Substitution gives (16). One obtains (17) from

$$\frac{p}{p_f} = \frac{v'(Q/\sum_{i=1}^{n}\hat{k}_i)}{v'(Q_f/\sum_{i=1}^{n}k_i)} = \left(\frac{Q}{Q_f}\frac{\sum_{i=1}^{n}k_i}{\sum_{i=1}^{n}\hat{k}_i}\right)^{\frac{-1}{e}} = \left(\frac{Q_f}{Q}\frac{\sum_{i=1}^{n}\hat{k}_i}{\sum_{i=1}^{n}k_i}\right)^{\frac{1}{e}}$$

$$= \left[\left(\frac{\sum_{i=1}^{n}k_i}{\sum_{i=1}^{n}\hat{k}_i}\right)^{\frac{h}{e+h}}\left(\frac{\sum_{i=1}^{n}g_i}{\sum_{i=1}^{n}\hat{g}_i}\right)^{\frac{e}{e+h}}\frac{\sum_{i=1}^{n}\hat{k}_i}{\sum_{i=1}^{n}k_i}\right]^{\frac{1}{e}} = \left[\left(\frac{\sum_{i=1}^{n}g_i}{\sum_{i=1}^{n}\hat{g}_i}\right)\left(\frac{\sum_{i=1}^{n}\hat{k}_i}{\sum_{i=1}^{n}k_i}\right)\right]^{\frac{1}{e+h}}$$

$$= \left[\frac{\sum_{i=1}^{n}\boldsymbol{s}_i\left(1+\frac{s_i-\boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{-h}}{\sum_{i=1}^{n}s_i\left(1+\frac{s_i-\boldsymbol{s}_i}{\boldsymbol{e}(1-s_i)+\boldsymbol{h}(1-\boldsymbol{s}_i)}\right)^{e}}\right]^{\frac{1}{e+h}}.$$

Q.E.D.

Analysis of Section 4:

Using the market calculations (20) and (21), rewrite profits to obtain

$$\boldsymbol{p}_i = r(Q)q_i - k_i w\left(q_i/k_i\right) - \boldsymbol{g}_i c\left(x_i/\boldsymbol{g}_i\right) - p(q_i - x_i)$$

$$= (r(Q)-p)q_i - k_i w\left(q_i/k_i\right) + px_i - \boldsymbol{g}_i c\left(x_i/\boldsymbol{g}_i\right)$$

$$= w'\left(Q/K\right)q_i - k_i w\left(q_i/k_i\right) + c'\left(Q/\Gamma\right)x_i - c\left(x_i/\boldsymbol{g}_i\right)$$

$$= w'\left(Q/K\right)\frac{\hat{k}_i}{K}Q - k_i w\left(\frac{\hat{k}_i}{K}\frac{Q}{k_i}\right) + c'\left(Q/\Gamma\right)\frac{\hat{g}_i}{\Gamma}Q - \boldsymbol{g}_i c\left(\frac{\hat{g}_i}{\Gamma}\frac{Q}{\boldsymbol{g}_i}\right).$$

The equilibrium quantity is given by

$$r(Q) - w'(Q/K) - c'(Q/\Gamma) = 0.$$

From this equation, and applying (22), it is a routine computation to show:

$$\frac{K}{Q}\frac{dQ}{dK} = \frac{K}{Q}\frac{-(w'')Q/K^2}{r' - w''/K - c''/\Gamma} = \frac{(r-p)\boldsymbol{b}^{-1}}{r\boldsymbol{a}^{-1} + (r-p)\boldsymbol{b}^{-1} + p\boldsymbol{h}^{-1}} = \frac{B}{A+B+C}.$$

Similarly,

$$\frac{\Gamma}{Q}\frac{dQ}{d\Gamma} = \frac{C}{A+B+C}.$$

Differentiating π_i, and using the analogous notation for

$$0 = \frac{K}{Q}\frac{\partial \pi_i}{\partial \hat{k}_i} = (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) - w'' \frac{s_i Q}{K} + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} + (w'' s_i \frac{Q}{K} + c'' \sigma_i \frac{Q}{\Gamma})\frac{K}{Q}\frac{dQ}{dK}$$

$$= (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} - \beta^{-1} w' s_i + (\beta^{-1} w' s_i + \eta^{-1} c' \sigma_i)\frac{K}{Q}\frac{dQ}{dK}$$

$$= (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} - r\left(Bs_i - (Bs_i + C\sigma_i)\frac{K}{Q}\frac{dQ}{dK} \right)$$

$$= (w' - w_i')(1 - s_i + s_i \frac{K}{Q}\frac{dQ}{dK}) + (c' - c_i')\sigma_i \frac{K}{Q}\frac{dQ}{dK} - r\left(Bs_i \left(1 - \frac{K}{Q}\frac{dQ}{dK}\right) - C\sigma_i \frac{K}{Q}\frac{dQ}{dK} \right)$$

Similarly, and symmetrically,

$$0 = \frac{\Gamma}{Q}\frac{\partial \pi_i}{\partial \hat{\gamma}_i} = (w' - w_i')s_i \frac{\Gamma}{Q}\frac{dQ}{d\Gamma} + (c' - c_i')(1 - \sigma_i + \sigma_i \frac{\Gamma}{Q}\frac{dQ}{d\Gamma}) - r\left(C\sigma_i \left(1 - \frac{\Gamma}{Q}\frac{dQ}{d\Gamma}\right) - Bs_i \frac{\Gamma}{Q}\frac{dQ}{d\Gamma} \right).$$

These equations can be expressed, substituting the elasticities with respect to capacity, as

$$\begin{bmatrix} A+B+C - s_i(A+C) & B\sigma_i \\ Cs_i & A+B+C - \sigma_i(A+B) \end{bmatrix} \begin{pmatrix} (w' - w_i')/r \\ (c' - c_i')/r \end{pmatrix} = \begin{pmatrix} Bs_i(A+C) - BC\sigma_i \\ c\sigma_i(A+B) - BCs_i \end{pmatrix}$$

The determinant of the left hand side matrix is given by

$$DET = [A+B+C - s_i(A+C)][A+B+C - \sigma_i(A+B)] - BCs_i\sigma_i$$

$$= (A+B+C)[(A+B+C)(1-s_i)(1-\sigma_i) + Bs_i(1-\sigma_i) + C(1-s_i)\sigma_i]$$

$$= (A+B+C)[A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)].$$

Thus,

$$\begin{pmatrix} (w'-w_i')/r \\ (c'-c_i')/r \end{pmatrix} = \frac{1}{DET}\begin{bmatrix} A+B+C-\sigma_i(A+B) & -B\sigma_i \\ -Cs_i & A+B+C-s_i(A+C) \end{bmatrix}\begin{pmatrix} Bs_i(A+C)-BC\sigma_i \\ c\sigma_i(A+B)-BCs_i \end{pmatrix}$$

$$= \frac{1}{DET}\begin{pmatrix} (A+B+C-\sigma_i(A+B))(Bs_i(A+C)-BC\sigma_i)-B\sigma_i(c\sigma_i(A+B)-BCs_i) \\ -Cs_i(Bs_i(A+C)-BC\sigma_i)+(A+B+C-s_i(A+C))(c\sigma_i(A+B)-BCs_i) \end{pmatrix}$$

$$= \frac{A+B+C}{DET}\begin{pmatrix} (Bs_i(A+C)-BC\sigma_i)-ABs_i\sigma_i \\ (C\sigma_i(A+B)-BCs_i)-ACs_i\sigma_i \end{pmatrix} = \frac{A+B+C}{DET}\begin{pmatrix} B[C(s_i-\sigma_i)+As_i(1-\sigma_i)] \\ C[B(\sigma_i-s_i)+A\sigma_i(1-s_i)] \end{pmatrix}$$

Thus,

$$MHI = \sum_{i=1}^{n}\left(\frac{(r(Q)-p-w_i')s_i+(p-c_i')\sigma_i}{r}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{(w'-w_i')s_i+(c'-c_i')\sigma_i}{r}\right)$$

$$= \sum_{i=1}^{n}\left[\left(\frac{A+B+C}{DET}\right)\left(s_iB[C(s_i-\sigma_i)+As_i(1-\sigma_i)]+\sigma_iC[B(\sigma_i-s_i)+A\sigma_i(1-s_i)]\right)\right]$$

$$= \sum_{i=1}^{n}\left[\left(\frac{A+B+C}{DET}\right)\left[BC(s_i-\sigma_i)^2+ABs_i^2(1-\sigma_i)+AC\sigma_i^2(1-s_i)\right]\right]$$

$$= \sum_{i=1}^{n}\frac{BC(s_i-\sigma_i)^2+ABs_i^2(1-\sigma_i)+AC\sigma_i^2(1-s_i)}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)}.$$

Q.E.D.

Derivation of the optimal quantity formula with constant elasticities

$$\begin{pmatrix} w'-w_i' \\ c'-c_i' \end{pmatrix} = r\begin{pmatrix} \dfrac{B[C(s_i-\sigma_i)+As_i(1-\sigma_i)]}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \\ \dfrac{C[B(\sigma_i-s_i)+A\sigma_i(1-s_i)]}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \end{pmatrix} \equiv r\begin{pmatrix} \psi_i \\ \chi_i \end{pmatrix}.$$

Then

$$\left(\frac{s_i Q}{k_i}\right)^{\frac{1}{b}} = w_i' = w' - r\boldsymbol{y}_i = r(1-\boldsymbol{q}-\boldsymbol{y}_i)$$

or,

$$k_i = s_i Q[r(1-\boldsymbol{q}-\boldsymbol{y}_i)]^{-b}$$

Similarly,

$$\boldsymbol{g}_i = \boldsymbol{s}_i Q[r(\boldsymbol{q}-\boldsymbol{c}_i)]^{-h}$$

The efficient solution satisfies $r - w'\left(Q_f \big/ \sum k_i\right) - c'\left(Q_f \big/ \sum \boldsymbol{g}_i\right) = 0$, or

$$0 = Q_f^{-\frac{1}{a}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{b}} r\left(\sum_{i=1}^{n} s_i[1-\boldsymbol{q}-\boldsymbol{y}_i]^{-b}\right)^{\frac{-1}{b}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{h}} r\left(\sum_{i=1}^{n} \boldsymbol{s}_i[\boldsymbol{q}-\boldsymbol{c}_i]^{-h}\right)^{\frac{-1}{h}}$$

Thus, substituting and dividing by $r=Q^{-1/\alpha}$

$$0 = \left(\frac{Q_f}{Q}\right)^{\frac{-1}{a}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{b}} \left(\sum_{i=1}^{n} s_i[1-\boldsymbol{q}-\boldsymbol{y}_i]^{-b}\right)^{\frac{-1}{b}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{h}} \left(\sum_{i=1}^{n} \boldsymbol{s}_i[\boldsymbol{q}-\boldsymbol{c}_i]^{-h}\right)^{\frac{-1}{h}}$$

or

$$1 = \left(\frac{Q_f}{Q}\right)^{A+\frac{1}{b}} \left(\sum_{i=1}^{n} s_i[1-\boldsymbol{q}-\boldsymbol{y}_i]^{-b}\right)^{\frac{-1}{b}} + \left(\frac{Q_f}{Q}\right)^{A+\frac{1}{h}} \left(\sum_{i=1}^{n} \boldsymbol{s}_i[\boldsymbol{q}-\boldsymbol{c}_i]^{-h}\right)^{\frac{-1}{h}}.$$

or

$$1 = \left(\frac{Q}{Q_f}\right)^{-\left(A+\frac{1}{b}\right)} \left(\sum_{i=1}^{n} s_i[1-\boldsymbol{q}-\boldsymbol{y}_i]^{-b}\right)^{\frac{-1}{b}} + \left(\frac{Q}{Q_f}\right)^{-\left(A+\frac{1}{h}\right)} \left(\sum_{i=1}^{n} \boldsymbol{s}_i[\boldsymbol{q}-\boldsymbol{c}_i]^{-h}\right)^{\frac{-1}{h}}.$$

This equation can be solved for the ratio $Q_f \big/ Q$ which yields the underproduction.

Analysis with a Competitive Fringe

In many circumstances, there are firms that are best modeled as price-takers. Moreover, in some instances, the competitive fringe may use a distinct production technology, and therefore have different cost elasticities. This section replicates the general model with a competitive fringe. Variables associated with the fringe are denoted with the subscript 0. For example, q_0 is the downstream quantity of the fringe, and β_0 the fringe's retailing cost elasticity. We will only consider the constant elasticity case here, although the more general model follows directly.

The efficient solution satisfies:

$$0 = Q^{-1/\alpha} - \left(\frac{Q-q_0}{K}\right)^{1/\beta} - \left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta},$$

$$\left(\frac{q_0}{k_0}\right)^{\frac{1}{\beta_0}} = \left(\frac{Q-q_0}{K}\right)^{1/\beta} \text{ and } \left(\frac{x_0}{\gamma_0}\right)^{\frac{1}{\eta_0}} = \left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta}$$

Thus

$$\left(\frac{Q-q_0}{K}\right) = \left(\frac{q_0}{k_0}\right)^{\frac{\beta}{\beta}}$$

Therefore,

$$0 = \frac{dQ}{K} - \frac{Q-q_0}{K^2}dK - dq_0\left(\frac{1}{K} + \frac{\beta}{\beta_0}\left(\frac{q_0}{k_0}\right)^{\beta/\beta_0 - 1}\frac{1}{k_0}\right)$$

$$= \frac{dQ}{K} - \frac{Q-q_0}{K^2}dK - dq_0\left(\frac{1}{K} + \frac{\beta}{\beta_0}\frac{1}{q_0}\frac{Q-q_0}{K}\right)$$

$$dq_0 = \frac{dQ - \frac{Q-q_0}{K}dK}{1 + \frac{\beta}{\beta_0}\frac{Q-q_0}{q_0}} = \frac{dQ - \frac{Q-q_0}{K}dK}{1 + \frac{\beta}{\beta_0}\frac{1-s_0}{s_0}}.$$

Similarly,

$$dx_0 = \frac{dQ - \dfrac{Q-x_0}{\Gamma}\,d\Gamma}{1+\dfrac{\boldsymbol{h}}{\boldsymbol{h}_0}\dfrac{1-\boldsymbol{s}_0}{\boldsymbol{s}_0}}.$$

Differentiating the price=marginal cost equation,

$$0 = dQ\left[-\frac{1}{\boldsymbol{a}}Q^{-1/\boldsymbol{a}-1} - \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}-1}\frac{1}{K} - \frac{1}{\boldsymbol{h}}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\boldsymbol{h}-1}\frac{1}{\Gamma}\right]$$

$$+\frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}-1}\frac{dq_0}{K} + \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}-1}\frac{Q-q_0}{K^2}\,dK + \frac{1}{\boldsymbol{h}}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\boldsymbol{h}-1}\frac{dx_0}{\Gamma}$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\boldsymbol{a}}Q^{-1/\boldsymbol{a}} - \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}}\frac{Q}{Q-q_0} - \frac{1}{\boldsymbol{h}}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\boldsymbol{h}}\frac{Q}{Q-x_0}\right]$$

$$+\frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}}\frac{1}{Q-q_0}\,\frac{dQ-\dfrac{Q-q_0}{K}\,dK}{1+\dfrac{\boldsymbol{b}}{\boldsymbol{b}_0}\dfrac{1-s_0}{s_0}} + \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}}\frac{dK}{K}$$

$$+\frac{1}{\boldsymbol{h}}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\boldsymbol{h}}\frac{1}{Q-x_0}\,\frac{dQ}{1+\dfrac{\boldsymbol{h}}{\boldsymbol{h}_0}\dfrac{1-\boldsymbol{s}_0}{\boldsymbol{s}_0}}$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\boldsymbol{a}}Q^{-1/\boldsymbol{a}} - \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}}\frac{Q}{Q-q_0}\left(1-\frac{1}{1+\dfrac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}_0 s_0}}\right)\right]$$

$$+\frac{dQ}{Q}\left[-\frac{1}{\boldsymbol{h}}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\boldsymbol{h}}\frac{Q}{Q-x_0}\left(1-\frac{1}{1+\dfrac{\boldsymbol{h}}{\boldsymbol{h}_0}\dfrac{1-\boldsymbol{s}_0}{\boldsymbol{s}_0}}\right)\right] + \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/\boldsymbol{b}}\frac{dK}{K}\left(1-\frac{1}{1+\dfrac{\boldsymbol{b}}{\boldsymbol{b}_0}\dfrac{1-s_0}{s_0}}\right)$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\boldsymbol{a}}Q^{-1/a} - \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/b}\frac{1}{1-s_0}\left(\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}_0 s_0+\boldsymbol{b}(1-s_0)}\right)\right]$$

$$+ \frac{dQ}{Q}\left[-\frac{1}{\boldsymbol{h}}\left(\frac{Q-x_0}{\Gamma}\right)^{1/h}\frac{1}{1-s_0}\left(\frac{\boldsymbol{h}(1-\boldsymbol{s})}{\boldsymbol{h}(1-\boldsymbol{s}_0)+\boldsymbol{h}_0\boldsymbol{s}_0}\right)\right] + \frac{1}{\boldsymbol{b}}\left(\frac{Q-q_0}{K}\right)^{1/b}\frac{dK}{K}\left(\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}_0 s_0+\boldsymbol{b}(1-s_0)}\right)$$

$$= -\frac{dQ}{Q}\left[Ar + \frac{(1-\boldsymbol{q})r}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0} + \frac{\boldsymbol{q}r}{\boldsymbol{h}(1-\boldsymbol{s}_0)+\boldsymbol{h}_0\boldsymbol{s}_0}\right] + (1-s_0)\frac{dK}{K}\frac{(1-\boldsymbol{q})r}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}$$

Define

$$A = \frac{1}{\boldsymbol{a}}, \quad B = \frac{1-\boldsymbol{q}}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}, \quad C = \frac{\boldsymbol{q}}{\boldsymbol{h}(1-\boldsymbol{s}_0)+\boldsymbol{h}_0\boldsymbol{s}_0}$$

Then,

$$\frac{K}{Q}\frac{dQ}{dK} = (1-s_0)\frac{B}{A+B+C}.$$

$$\frac{d}{dK}\frac{Q-q_0}{K} = \frac{dQ-dq_0}{K} - \frac{Q-q_0}{K^2}dK = \frac{1}{K}\left[dQ - \frac{dQ-\frac{Q-q_0}{K}dK}{1+\frac{\boldsymbol{b}}{\boldsymbol{b}_0}\frac{1-s_0}{s_0}} - \frac{Q-q_0}{K}dK\right]$$

$$= \frac{1}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}\right]\left(dQ - \frac{Q-q_0}{K}dK\right) = \frac{dK}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}\right]\left(\frac{dQ}{dK} - \frac{Q-q_0}{K}\right)$$

$$= \frac{dK}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}\right]\left(\frac{K}{Q}\frac{dQ}{dK} - \frac{Q-q_0}{Q}\right)\frac{Q}{K} = \frac{dK}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}\right]\left(\frac{(1-s_0)B}{A+B+C} - (1-s_0)\right)\frac{Q}{K}$$

$$= \frac{dK}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}\right]\left(\frac{B}{A+B+C} - 1\right)\frac{Q-q_0}{K} = \frac{dK}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0)+\boldsymbol{b}_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}$$

$$\frac{d}{dK}\frac{Q-x_0}{\Gamma} = \frac{1}{\Gamma}\left[\frac{dQ}{dK} - \frac{dQ/dK}{1+\frac{\boldsymbol{h}}{\boldsymbol{h}_0}\frac{1-\boldsymbol{s}_0}{\boldsymbol{s}_0}}\right] = \frac{1}{\Gamma}\frac{\boldsymbol{h}(1-\boldsymbol{s}_0)}{\boldsymbol{h}(1-\boldsymbol{s}_0)+\boldsymbol{h}_0\boldsymbol{s}_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

The firm’s profits are

$$\boldsymbol{p} = r(Q)\frac{\hat{k}_i}{K}(Q-q_0) - k_i w\left(\frac{\hat{k}_i}{K}\frac{(Q-q_0)}{k_i}\right) - \boldsymbol{g}_i c\left(\frac{\hat{\boldsymbol{g}}_i}{\Gamma}\frac{Q-x_0}{\boldsymbol{g}_i}\right) + p\left(\frac{\hat{\boldsymbol{g}}_i}{\Gamma}(Q-x_0) - \frac{\hat{k}_i}{K}(Q-q_0)\right)$$

$$= w'\left(\frac{Q-q_0}{K}\right)\frac{\hat{k}_i}{K}(Q-q_0) - k_i w\left(\frac{\hat{k}_i}{K}\frac{Q-q_0}{k_i}\right) + c'\left(\frac{Q-x_0}{\Gamma}\right)\frac{\hat{\boldsymbol{g}}_i}{\Gamma}(Q-x_0) - \boldsymbol{g}_i c\left(\frac{\hat{\boldsymbol{g}}_i}{\Gamma}\frac{Q-x_0}{\boldsymbol{g}_i}\right)$$

$$\frac{\partial \boldsymbol{p}}{\partial \hat{k}_i} = (w' - w_i')\frac{\partial}{\partial \hat{k}_i}\frac{\hat{k}_i(Q-q_0)}{K} + w''\left(\frac{d}{dK}\frac{Q-q_0}{K}\right)\frac{\hat{k}_i(Q-q_0)}{K}$$
$$+ (c' - c_i')\frac{\partial}{\partial \hat{k}_i}\frac{\hat{g}_i(Q-x_0)}{\Gamma} + \left(c''\right)\left(\frac{\hat{g}_i}{\Gamma}\frac{d}{dK}(Q-x_0)\right)\frac{\hat{g}_i}{\Gamma}(Q-x_0)$$

$$= (w' - w_i')\left(\frac{(Q-q_0)}{K} - \frac{\hat{k}_i}{K}\left(\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0) + \boldsymbol{b}_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}\right)\right)$$

$$+ w''\left(\frac{1}{K}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0) + \boldsymbol{b}_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}\right)\frac{\hat{k}_i(Q-q_0)}{K}$$

$$+ (c' - c_i')\frac{\hat{g}_i}{\Gamma}\frac{h(1-s_0)}{h(1-s_0) + h_0 s_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

$$+ c''\frac{\hat{g}_i}{\Gamma}\frac{h(1-s_0)}{h(1-s_0) + h_0 s_0}\frac{Q-q_0}{K}\frac{B}{A+B+C})\frac{\hat{g}_i}{\Gamma}(Q-x_0)$$

$$= (w' - w_i')\frac{(Q-q_0)}{K}\left(1 - \frac{s_i}{1-s_0}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0) + \boldsymbol{b}_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)$$

$$+ \frac{w'}{\boldsymbol{b}}\left[\frac{\boldsymbol{b}(1-s_0)}{\boldsymbol{b}(1-s_0) + \boldsymbol{b}_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{s_i}{1-s_0}\frac{Q-q_0}{K}$$

$$+ (c' - c_i')\frac{s_i}{1-s_0}\frac{h(1-s_0)}{h(1-s_0) + h_0 s_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

$$+ \frac{c'}{h}\frac{s_i}{1-s_0}\frac{h(1-s_0)}{h(1-s_0) + h_0 s_0}\frac{Q-q_0}{K}\frac{B}{A+B+C})$$

Thus,

$$0=(w'-w_i')\left(1-\left[\frac{bs_i}{b(1-s_0)+b_0s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)+w'\left[\frac{s_i}{b(1-s_0)+b_0s_0}\right]\left(\frac{A+C}{A+B+C}\right)$$

$$+(c'-c_i')\frac{hs_i}{h(1-s_0)+h_0s_0}\frac{B}{A+B+C}+c'\frac{s_i}{h(1-s_0)+h_0s_0}\frac{B}{A+B+C}$$

Thus,

$$0=\frac{(w'-w_i')}{r}\left(1-\left[\frac{bs_i}{b(1-s_0)+b_0s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)+(1-q)\left[\frac{s_i}{b(1-s_0)+b_0s_0}\right]\left(\frac{A+C}{A+B+C}\right)$$

$$+\frac{c'-c'}{r}{}_i\,\frac{hs_i}{h(1-s_0)+h_0s_0}\frac{B}{A+B+C}+q\,\frac{s_i}{h(1-s_0)+h_0s_0}\frac{B}{A+B+C}$$

Employing the definitions $y_i=\frac{w'-w_i'}{r}$, $c_i=\frac{c'-c_i'}{r}$ and using the analogous equation from the optimization of $\hat{g}_i$ we have

$$\begin{bmatrix} A+B+C-\dfrac{b(A+C)s_i}{b(1-s_0)+b_0s_0} & \dfrac{hBs_i}{h(1-s_0)+h_0s_0} \\ \dfrac{bCs_i}{b(1-s_0)+b_0s_0} & A+B+C-\dfrac{h(A+B)s_i}{h(1-s_0)+h_0s_0} \end{bmatrix}\begin{pmatrix} y_i \\ c_i \end{pmatrix}=\begin{pmatrix} B(A+C)s_i-BCs_i \\ C(A+B)s_i-BCs_i \end{pmatrix}$$

Let $\bar{b}=b(1-s_0)+b_0s_0$ and $\bar{h}=h(1-s_0)+h_0s_0$. Then the determinant of the LHS matrix is

$$DET=(A+B+C)^2-(A+B+C)\left[\frac{b}{\bar{b}}(A+C)s_i+\frac{h}{\bar{h}}(A+B)s_i\right]+\frac{b}{\bar{b}}\frac{h}{\bar{h}}\left[(A+C)(A+B)-BC\right]s_is_i$$

$$=(A+B+C)\left[(A+B+C)-\left[\frac{b}{\bar{b}}(A+C)s_i+\frac{h}{\bar{h}}(A+B)s_i\right]+\frac{b}{\bar{b}}\frac{h}{\bar{h}}As_is_i\right]$$

$$=(A+B+C)\left[A\left(1-\frac{b}{\bar{b}}s_i\right)\left(1-\frac{h}{\bar{h}}s_i\right)+B\left(1-\frac{h}{\bar{h}}s_i\right)+C\left(1-\frac{b}{\bar{b}}s_i\right)\right]$$

Thus

$$\begin{pmatrix}\psi_i \\ \chi_i\end{pmatrix} = \frac{1}{DET}\begin{bmatrix} A+B+C-\dfrac{\eta(A+B)\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} & \dfrac{-\eta B\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \\ \dfrac{-\beta C s_i}{\beta(1-s_0)+\beta_0 s_0} & A+B+C-\dfrac{\beta(A+C)s_i}{\beta(1-s_0)+\beta_0 s_0}\end{bmatrix}\begin{pmatrix} B(A+C)s_i - BC\sigma_i \\ C(A+B)\sigma_i - BCs_i\end{pmatrix}$$

Thus,

$$\psi_i = \frac{B}{DET}\left[\left(A+B+C-\frac{\eta}{\bar{\eta}}(A+B)\sigma_i\right)\left((A+C)s_i - C\sigma_i\right) - \frac{\eta}{\bar{\eta}}\sigma_i\left(C(A+B)\sigma_i - BCs_i\right)\right]$$

$$= \frac{B}{DET}\left[(A+B+C)\left((A+C)s_i - C\sigma_i\right) - \frac{\eta}{\bar{\eta}}(A+B)\sigma_i\left((A+C)s_i\right) - \frac{\eta}{\bar{\eta}}\sigma_i\left(-BCs_i\right)\right]$$

$$= \frac{B}{DET}\left[(A+B+C)\left((A+C)s_i - C\sigma_i\right) - \frac{\eta}{\bar{\eta}}(A+B+C)As_i\sigma_i\right]$$

$$= \frac{B(A+B+C)}{DET}\left[\left((A+C)s_i - C\sigma_i\right) - A\frac{\eta}{\bar{\eta}}s_i\sigma_i\right] = \frac{B(A+B+C)}{DET}\left[C(s_i-\sigma_i) + As_i\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right)\right]$$

$$= \frac{B\left[C(s_i-\sigma_i) + As_i\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right)\right]}{A\left(1-\frac{\beta}{\bar{\beta}}s_i\right)\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + B\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + C\left(1-\frac{\beta}{\bar{\beta}}s_i\right)}$$

Analogously,

$$\chi_i = \frac{C\left[B(\sigma_i - s_i) + A\sigma_i\left(1-\frac{\beta}{\bar{\beta}}s_i\right)\right]}{A\left(1-\frac{\beta}{\bar{\beta}}s_i\right)\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + B\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + C\left(1-\frac{\beta}{\bar{\beta}}s_i\right)}$$

Computations are performed as follows. First, note that

$$\left(\frac{s_i Q}{k_i}\right)^{\frac{1}{b}} = w_i' = w' - r\psi_i = r(1-\theta-\psi_i)$$

or,

$$k_i = s_i Q[r(1-\boldsymbol{q}-\boldsymbol{y}_i)]^{-b}$$

Similarly,

$$\boldsymbol{g}_i = \boldsymbol{s}_i Q[r(\boldsymbol{q}-\boldsymbol{c}_i)]^{-h}$$

Choose price and quantity units so that the initial price and quantity are both unity. Then capacities are readily computed from these equations, and are correct up to a scalar. The price is generally $r = Q^{-A}$, and thus the effect of a change in the allocation of the capacities can be computed by solving for the market shares the equations

$$F = k_i^{1/b}(1-\boldsymbol{q}-\boldsymbol{y}_i) - s_i^{1/b} Q^{1/b + A} = 0$$

and $G = \boldsymbol{g}_i^{1/h}(\boldsymbol{q}-\boldsymbol{c}_i) - \boldsymbol{s}_i^{1/h} Q^{1/h + A} = 0$

The strategy for computation is to guess values of Q and θ, then see if the firms' desired shares, as given by the solutions to F=G=0, sum to one. F and G have the useful properties that they are both positive at s_i=σ_i=0, F is increasing in σ_i and decreasing in s_i, and G is decreasing in σ_i and increasing in s_i. Thus, a search starting at (0,0) and always increasing in both arguments finds a solution for the desired shares although the solution may be the upper bound of 1. These solutions provide a sum of shares; the two equations

(*) $$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \boldsymbol{s}_i = 1$$

are then used to compute the solution for Q and θ, the remaining unknowns. The unknowns are bounded, θ in [0,1] and Q greater than zero and no greater than the efficient quantity. The latter is calculable directly from the capacities. The desired shares are decreasing in the total quantity Q, guaranteeing a unique solution in Q for any given θ. Generally, as θ rises, the desired values of σ_i rise and s_i fall, but we don't have a proof for this. Mathematica 3.0 invariably finds a solution while Mathematica 2.2. did not.