Handout #6: Technical Supplement on Milgrom-Weber Auction Theory

Complementarity

Let $x \wedge y$, $x \vee y$, refer to the component-wise minima (*x meet y*) and maxima (*x join y*), respectively.

A function f:R^n→R is supermodular if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$.

Remark: If f is twice-differentiable, then supermodularity reduces to: $i \neq j$ *implies* $\dfrac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$.

This, in turn, is equivalent to "increasing differences." That is, for $x_i > y_i$,

$f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)$ is nondecreasing in x_j for $j \neq i$.

If f is a payoff function, the variables of f are said to be complementary.

Affiliation

If the function log f is supermodular, f is said to be log supermodular [log-spm]. If f is a density, then the random variables with density f are said to be *affiliated*. If there are two of them, f is said to have the *monotone likelihood ratio property* (MLRP).

(i) Affiliation is equivalent to the statement that $E[\alpha(\mathbf{X})|a_i \leq X_i \leq b_i]$ is nondecreasing in a_i, b_i for all nondecreasing functions α.

Proof: Consider $\varphi(y)$=E[$\alpha(x,Y)|Y=y, a \leq x \leq b$]. Below, expectations refer to conditioning on $Y=y$, $a \leq x \leq b$.

$$\varphi'(y) = E\left[\frac{\partial \alpha}{\partial y}\right] + \int \alpha(x,y)\left(\frac{f_y(s\,|\,y)}{\int_a^b f(z\,|\,y)dz} - \frac{f(x\,|\,y)\int_a^b f_y(z\,|\,y)dz}{\left(\int_a^b f(z\,|\,y)dz\right)^2}\right)dx$$

$$= E\left[\frac{\partial \alpha}{\partial y}\right] + E\left[\alpha(x,y)\frac{f_y(x\,|\,y)}{f(x\,|\,y)}\right] - E[\alpha(x,y)]\,E\left[\frac{f_y(x\,|\,y)}{f(x\,|\,y)}\right] = E\left[\frac{\partial \alpha}{\partial y}\right] + COV(\alpha, \frac{f_y}{f}).$$

If the MLRP is satisfied, this is nonnegative. Conversely, let α be increasing in x and constant in y. Then φ is nondecreasing for all y, a, and b if and only if the MLRP is satisfied.

(ii) Nondecreasing functions of affiliated r.v.'s are affiliated (see Milgrom-Weber).

Let x, y have density $f(x,y)$, and denote the density of y given x by $f_Y(y|x)$, with cdf $F_Y(y|x)$.

(iii) $F_Y(y|x)$ is nonincreasing in x (First Order Stochastic Dominance).

The characteristic function of a set, 1_A, is the function which is 1 if $x \in A$ and 0 otherwise. Note $\Pr[X_i \geq x_i] = E[1_{\{X_i \geq x_i\}}]$. It follows that $\Pr[X_i \geq x_i \mid X_j = x_j]$ is nondecreasing in x_j.

(iv) f is log-spm if and only if f_Y is log-spm.

Proof: $\frac{\partial^2}{\partial x\ \partial y} \log\ f_Y(y|x) = \frac{\partial^2}{\partial x\ \partial y} \log \left(\frac{f(x,y)}{\int f(x,z)\,dz} \right) =$

$$\frac{\partial^2}{\partial x\,\partial y} \log\ f(x,y) - \frac{\partial^2}{\partial x\ \partial y} \log \left(\int f\ (x,z)\ dz \right) = \frac{\partial^2}{\partial x\,\partial y} \log f(x,y)$$

(v) Independently distributed random variables are affiliated.

(vi) If $f(y|x)$ is log-spm, $F(y|x)$ is log-spm

Proof: $\frac{\partial}{\partial x} \frac{f(y|x)}{F(y|x)} = \frac{f_2(y|x)}{F\ (y|x)} - \frac{f(y|x)\,F_2(y|x)}{F(y|x)^2}$

$$= \frac{f(y|x)}{F(y|x)^2} \left[\frac{f_2(y|x)}{f(y|x)} F(y|x) - F_2(y|x) \right]$$

$$= \frac{f\ (y|x)}{F(y|x)^2} \left[\int_0^y \left(\frac{f_2(y|x)}{f\ (y|x)} - \frac{f_2(z|x)}{f\ (z|x)} \right) f(z|x)\,dz \right] \geq 0.$$

(vii) if f,g are log-spm, fg is log-spm. Proof is $\log fg = \log f + \log g$.

Auction Environment

Bidder i privately receives a signal that is the realization of the r.v. X_i; the vector $(X_1,\ldots,X_n,\boldsymbol{S})$ are affiliated and the X_i's are symmetrically distributed. The payoff to bidder i is $u(X_i,X_{-i},\boldsymbol{S})$. u is assumed nondecreasing in all arguments. We fix attention on bidder 1 and let $Y = \max\ \{X_2,\ldots,X_n\}$. Y is affiliated with X_1. Let $f_Y(Y|x)$ be the density of Y given $X_1 = x$, with distribution function F_Y. Let $v(x,y) = E[u|x_1 = x, Y = y]$. Since u is nondecreasing, so is v.

Second Price Auction

In a second price auction, the high bidder obtains the object and pays the second highest bid.

A symmetric equilibrium bidding function is a function B_2 such that, given all other bidders bid according to B_2, the remaining bidder maximizes expected profit by bidding $B_2(x)$ given signal x. Consider bidder 1 with signal x who instead bids $B_2(z)$. This bidder earns

$$\pi = \int_0^z (v(x,y) - B_2(y)) f_Y(y \mid x) dy.$$

In order for B_2 to be an equilibrium, π must be maximized at $z=x$, which implies

$$B_2(x) = v(x,x).$$

It is straightforward to show that B_2 is indeed an equilibrium, and is the only symmetric equilibrium.

If a reserve price (minimum acceptable bid) r is imposed, bidders with signals below x_r, where $E[v(x_r, Y) | Y \le x_r] = r$, do not submit bids; otherwise the equilibrium is unperturbed. Note however, that the minimum submitted bid, $B_2(x_r) > r$!

Suppose the seller knows S_i. Should the seller tell the bidders S? The thought experiment is a commitment to honestly reveal the information prior to discovering it. Let

$$w(x,y,s) = E[u \mid X_1=x, Y=y, S_i=s].$$

$$\begin{aligned} v(y,y) &= E[w(X_1, Y, S_i) \mid X_1 = Y = y] \\ &= E[w(Y, Y, S_i) \mid X_1 = Y = y] \\ &\le E[w(Y, Y, S_i) \mid X_1 \ge Y = y]. \end{aligned}$$

The seller's revenue with no disclosure, R_N, is

$$\begin{aligned} R_N &= E[v(Y,Y) \mid X_1 \ge Y] \\ &\le E[E[w(Y,Y,S_i) \mid X_1 \ge Y] \mid x_1 > Y] \\ &= E[w(Y,Y,S_i) \mid X_1 > Y] = R_I, \end{aligned}$$

the revenue with disclosure of S_i.

That is, it always pay to reveal the information accurately.

First Price Auction

In a first price auction, the high bidder obtains the object and pays her bid. Suppose B_1 is a symmetric equilibrium. The profits to bidder 1, with signal x, who bids $B_1(z)$, are:

$$\pi = \int_0^z (v(x,y) - B(z)) f_Y(y \mid x) dy.$$

Maximizing with respect to z, and setting z=x, yields the first order differential equation

$$B_1'(x) = \frac{f_Y(x \mid x)}{F_Y(x \mid x)}(v(x,x) - B_1(x)).$$

Suppose that the reserve price is zero. Then the differential equation has solution

$$B_1(x) = \int_0^x e^{-\int_y^x \frac{f_Y(z|z)}{F_Y(z|z)}dz} \frac{f_Y(y \mid y)}{F_Y(y \mid y)} v(y,y) dy.$$

(If the reserve price $r > 0$, the screening level is x_r and B_1 satisfies $B_1(x_r)$=r.) Integrating $B_1(x)$ by parts and assuming v(0,0)=0, we have:

$$B_1(x) = v(x,x) - \int_0^x e^{-\int_y^x \frac{f_Y(z|z)}{F_Y(z|z)}dz} \left(\frac{d}{dy} v(y,y) \right) dy.$$

Conditional on winning with a signal of x (probability $F_Y(x|x)$), a bidder in a second price auction pays

$$EB_2 = \int_0^x v(y,y) \frac{f_Y(y \mid x)}{F_Y(x \mid x)} dy = v(x,x) - \int_0^x \frac{F_Y(y \mid x)}{F_Y(x \mid x)} \left[\frac{d}{dy} v(y,y) \right] dy.$$

Note that $\log F_Y(x \mid x) - \log F_Y(y \mid x) = \int_y^x \frac{f_Y(z \mid x)}{F_Y(z \mid x)} dz \geq \int_y^x \frac{f_Y(z \mid z)}{F_Y(z \mid z)} dz$ (by(vi))

and thus, $\frac{F_Y(y \mid x)}{F_Y(x \mid x)} \leq e^{-\int_y^x \frac{f_Y(z|z)}{F_Y(z|z)}dz}$.

Since v is nondecreasing, the expected payment by a winning bidder with signal x is higher in a second price auction than in a first price auction.