Handout #3: Peak Load Pricing

Consider a firm that experiences two kinds of costs – a capacity cost and a marginal cost. How should capacity be priced? This issue is applicable to a wide variety of industries, including pipelines, airlines, telephone networks, construction, electricity, highways, and the internet.

The basic peak-load pricing problem, pioneered by Marcel Boiteaux, considers two periods. The firm's profits are given by

$$\pi = p_1 q_1 + p_2 q_2 - \beta \max \{q_1, q_2\} - mc(q_1 + q_2).$$

Prices equal to marginal costs are not sustainable, because a firm selling with price equal to marginal cost would not earn a return on the capacity, and thus would lose money and go out of business. A capacity charge is necessary. The question of peak load pricing is where the capacity charge should be allocated.

Demands are ordinarily assumed independent, but this is neither a good assumption nor a necessary one. Our previous analysis suggests how the solution will change, however, and so I will stick with independent demands for simplicity.

Social welfare is

$$W = \int_0^{q_1} p_1(x)dx + \int_0^{q_{21}} p_2(x)dx - \beta \max \{q_1, q_2\} - mc(q_1 + q_2).$$

The Ramsey problem is to maximize W subject to a profit condition. As always, write the Lagrangian

$$L = W + \lambda\, \pi.$$

Therefore,

$$0 = \frac{\partial L}{\partial q_1} = p_1(q_q) - \beta 1_{q_1 \geq q_2} - mc + \lambda\left(p_1(q_q) + q_1 p_1'(q_1) - \beta 1_{q_1 \geq q_2} - mc \right)$$

Or,

$$\frac{p_1(q_1) - \beta 1_{q_1 \geq q_2} - mc}{p_1} = \frac{\lambda}{\lambda + 1} \frac{1}{\varepsilon_1}$$

where $1_{q_1 \geq q_2}$ is the characteristic function of the event $q_1 \geq q_2$.

Similarly,

$$\frac{p_2(q_2) - \beta 1_{q_1 \le q_2} - mc}{p_2} = \frac{\lambda}{\lambda + 1} \frac{1}{\varepsilon_2}$$

Note as before that $\lambda \to \infty$ yields the monopoly solution.

There are two potential types of solution. Let the demand for good 1 exceed the demand for good 2. Either $q_1 > q_2$, or the two are equal.

Case 1: $q_1 > q_2$.

$$\frac{p_1(q_1) - \beta - mc}{p_1} = \frac{\lambda}{\lambda + 1} \frac{1}{\varepsilon_1} \text{ and } \frac{p_2(q_2) - mc}{p_2} = \frac{\lambda}{\lambda + 1} \frac{1}{\varepsilon_2}.$$

In case 1, with all of the capacity charge allocated to good 1, quantity for good 1 still exceeds quantity for good 2. Thus, the peak period for good 1 is an extreme peak. In contrast, case 2 arises when assigning the capacity charge to good 1 would reverse the peak – assigning all of the capacity charge to good 1 would make period 2 the peak.

Case 2: $q_1 = q_2$.

The first order conditions become inequalities, of the form

$$0 \le p_1(q_q) - mc + \lambda\left(p_1(q_q) + q_1 p_1'(q_1) - mc\right) \le (1 + \lambda)\beta.$$

$$0 \le \frac{p_1(q_1) - mc}{p_1} - \frac{\lambda}{\lambda + 1} \frac{1}{\varepsilon_1} \le \beta \text{ and } 0 \le \frac{p_2(q_2) - mc}{p_2} - \frac{\lambda}{\lambda + 1} \frac{1}{\varepsilon_2} \le \beta.$$

These must solve at $q_1 = q_0 = q$. The profit equation can be written

$p_1(q) - mc + p_2(q) - mc = \beta$

This equation shows that the capacity charge is shared across the two markets proportional to the inverse demand.

Priority Pricing

The peak load problem is essentially a cost allocation problem. It has an efficiency aspect, in that pricing matters to relative demand, but that efficiency aspect is incorporated in a familiar way, using inverse elasticities. The priority pricing problem introduced by Robert Wilson has a superficial similarity to the peak load problem – when capacity is reached, who should be rationed? Implicitly, the peak load formulation implies a spot market, so that each market is rationed efficiently. In many circumstances, it is not possible to use prices *ex post* to ration the market. For example, absent smart appliances, it is difficult for homeowners to adjust electric demand in real time as prices vary – homeowners aren't even informed about the abrupt price changes. Priority pricing is a means of contracting in advance when capacity, or demand, is stochastic.

At this time, the problem of stochastic demand and priority pricing has not been adequately addressed. In particular, with stochastic demand, there is an issue of whether all customers are able to participate in the *ex ante* priority market.

Consider a case of a continuum of consumers, each of whom desires one unit. As will become clear, it doesn't matter if some consumers desire multiple units – each unit can be treated as demanded by a separate consumer. Rank the consumers by their valuations for the good, so that the q^{th} consumer has a value $p(q)$ for the good, and p is downward sloping.

The quantity available is a random variable with distribution F. Priority pricing is a charge schedule c which provides a unit to a customer paying $c(q)$ whenever realized supply is q or greater.

It is a straightforward exercise to calculate the incentive compatible c schedule. A customer of type q should choose to pay $c(q)$ for the q^{th} spot in the priority list. This leads to the incentive constraint:

$$u(q) = (p(q) - c(q))(1 - F(q)) \geq (p(q) - c(\hat{q}))(1 - F(\hat{q})).$$

The envelope theorem gives

$$u'(q) = p'(q)(1 - F(q)).$$

It is a straightforward exercise to demonstrate that the first order condition is sufficient; see handout #2. Let $F(H)=1$, so that $u(H)=0$. Then

$$(p(q) - c(q))(1 - F(q)) = u(q) = -\int_q^H u'(s)ds = -\int_q^H p'(s)(1 - F(s))ds = p(q)(1 - F(q)) - \int_q^H p(s)f(s)ds$$

Thus,

$$c(q) = \int_q^H p(s)\frac{f(s)}{1 - F(q)}ds = E[spot\ \ price \mid p(s) \geq p(q)].$$

Revenues to the firm from the priority pricing are

$$R = \int_0^H c(q)(1 - F(q))dq = \int_0^H \int_q^H p(s)f(s)\,ds\,dq = \int_0^H qp(q)f(q)\,dq.$$

This is the revenue associated with a competitive supply; a monopolist might have an incentive to withhold capacity to boost prices. How does a monopolist do so? Withholding of capacity has the property of changing the distribution of available supply, in a first order stochastic dominant manner. In particular, the monopolist can offer any distribution of capacity G, provided $G \geq F$. What is the monopolist's solution? Rewrite R to obtain

$$R = \int_0^H qp(q)g(q)dq = \int_0^H MR(q)(1 - G(q))dq.$$

Provided marginal revenue MR is single-peaked,

$$G = \begin{cases} F \text{ if } MR \geq 0 \\ \\ 1 \text{ if } MR < 0 \end{cases}.$$

That is, the monopolist cuts off the capacity at the monopoly supply.

Matching Problems

Priority pricing is a solution to a matching problem, matching the high value buyers with capacity. Many other problems have this feature, that it is desirable to match high types with high types and low types with low types. Such models have been used as models of marriage, employment, university admissions, incentive contracts, and other categories. Wilson examines not just the continuum matching, in which each probability of service interruption is separately priced, but also finite groups. Rather than offer a continuum of categories, consider offering just two – high priority service and low priority service. How well does such a priority service do?

The answer is remarkably well. Consider first the linear demand case with a uniform distribution of outages. Perfect matching gets a payoff

$$\int_0^1 p(q)(1-q)dq = \int_0^1 (1-q)^2 dq = \frac{1}{3}.$$

No matching – that is a random assignment – produces an expected value of ¼, a fact which is evident from

$$\int_0^1 p(q)dq \int_0^1 (1-q)dq = \left(\int_0^1 (1-q)dq \right)^2 = \frac{1}{4}.$$

Now consider two groups of equal size. The high value group has an average value of ¾, and is served with probability $\int_0^{1/2} 2q dq + \int_{1/2}^1 1 dq =$ ¾. The low value group has average value ¼ and is served with probability 1/4. Thus, the expected value from two categories is

$\frac{1}{2}\left(\frac{9}{16}+\frac{1}{16}\right)=\frac{5}{16}$. Note that 5/16 is 75% of the way from ¼ to 1/3! That is, a single group captures 75% of net value of a continuum of types!

The linear/uniform distribution is special; however, I show elsewhere that, provided a common hazard rate assumption is satisfied, two groups of distinguished by being above or below the mean generally captures 50% or more of the possible gains over no priority pricing. That is, even two classes is sufficient to capture a majority of the gains arising from priority pricing. Wilson shows that the losses from finite classes are on the order of $1/n^2$.