Handout #2: Revenue Maximization

The Two Type Model

In the two type case, assume there is a consumer L, for low type, with value $V_L(q)$ for quantity q, and H, for high type, with value v_H. Both value nothing at zero, so $v_L(0)=v_H(0)=0$. The high type is assumed to have higher demand for every positive quantity $q>0$:

(1) $\quad v'_H(q) \geq v'_L(q)$.

The monopolist offers two quantities q_L and q_H at prices R_L and R_H, respectively, targeted to the consumers L and H. In order for consumers to agree to purchase, two conditions known as individual rationality conditions must be satisfied

($\mathrm{IR_L}$) $\quad v_L(q_L)-R_L \geq 0$

($\mathrm{IR_H}$) $\quad v_H(q_H)-R_H \geq 0$.

Note that, rather than offer a plan in which the consumers don't participate, the monopolist could just as well offer $(q,R)=(0,0)$ and get the same outcome, in which case IR is satisfied. In addition, the monopolist must offer plans constructed so that L chooses (q_L,R_L) and H chooses (q_H,R_H). The conditions governing these plans are called *incentive compatibility conditions* and are mathematically formulated as follows.

($\mathrm{IC_L}$) $\quad v_L(q_L)-R_L \geq v_L(q_H)-R_H$

($\mathrm{IC_H}$) $\quad v_H(q_H)-R_H \geq v_H(q_L)-R_L$.

The condition ($\mathrm{IC_L}$) merely states that the utility the L consumer gets from purchasing the L plan is at least as great as if the L consumer purchases the H plan. Note that, if the monopolist had designed the plan so that the L consumer chose to purchase the H plan, he could have just as well offered the H plan to the L consumer in the first place, so that $\mathrm{IC_L}$ would hold. Thus, $\mathrm{IC_L}$ can be considered a constraint on the monopolist, and is without loss of generality. The $\mathrm{IC_H}$ constraint is analogous.

The monopolist is assumed to have a constant marginal cost c, and to maximize profit $R_L + R_H - c(q_L + q_H)$.

The analysis of the monopolist's behavior is performed by a series of claims, which will simplify the problem until a solution is obvious.

Claim 1: $q_L \leq q_H$.

Proof: Rearrange ICL and ICH to obtain

$$v_H(q_H) - v_H(q_L) \geq R_H - R_L \geq v_L(q_H) - v_L(q_L).$$

This gives

$$\int_{q_L}^{q_H} v'_H(q)\ dq = v_H(q_H) - v_H(q_L) \geq v_L(q_H) - v_L(q_L) = \int_{q_L}^{q_H} v'_L(q)\ dq,$$

or,

$$\int_{q_L}^{q_H} v'_H(q) - v'_L(q)\ dq \geq 0,$$

from which (1) proves the claim.

Claim 2: IR_H can be ignored. That is, IC_H and IR_L imply IR_H.

Proof: Using first IC_H then IR_L, note that

$$v_H(q_H) - R_H \geq v_H(q_L) - R_L = \int_0^{q_L} v'_H(q)\ dq - R_L \geq \int_0^{q_L} v'_L(q)\ dq - R_L = v_L(q_L) - R_L \geq 0.$$

Thus, if IC_H and IR_L are satisfied, then IR_H is automatically satisfied, and can be ignored.

Claim 3: IC_H is satisfied with equality at the monopolist's profit maximization.

Proof: Suppose not. Then the monopolist can increase R_H up to the point where IC_H is satisfied with equality, without violating either IR_L or IC_L. Since this increases revenue, the monopolist would do so, contradicting the assumption that the monopolist had maximized profit.

Claim 4: IR_L holds with equality.

Proof: Otherwise the monopolist could raise both R_L and R_H by the same amount, without violating the constraints.

Claims 3 and 4 let us express the monopolists objective function in terms of the quantities, merely by using the constraints that hold with equality. That is,

$$R_L + R_H - c(q_L+q_H) = 2v_L(q_L) + v_H(q_H) - v_L(q_H) - c(q_L+q_H).$$

This gives the first order conditions

$$0 = v'_H(q_H) - c,$$

and

$$0 = 2v_L'(q_L) - v_H'(q_L) - c.$$

The second may not be satisfiable, and in fact, if the demand of the high type is twice or more the demand of the low type, that is, $v_H'(q) > 2v_L'(q)$, then the monopolist's optimal quantity q_L=0, and the low type is shut out of the market.

We can deduce the following things from these equations and the IR and IC constraints.

1. The high type gets the "efficient" quantity (i.e. the quantity that a benevolent social planner would award him.

2. The low type gets strictly less than the efficient quantity.

3. The high type has a positive consumer surplus, that is, $v_H(q_H)-R_H>0$, unless $q_L=0$.

4. The low type gets zero consumer surplus.

The Continuum Model

Suppose consumers have utility $v(q,t)$ - p, where t is the type in [0,1] with density $f(t)$, q is quantity and p is the payment made. The monopolist will place a aggregate charge $R(q)$ for the purchase of q. What should the schedule of prices $R(q)$ look like?

Define the shadow price $p(q,t) = v_q(q,t)$, which gives the demand curve of the type t.

Assume $p_t(q,t) > 0$, that is, higher types have higher demands, and that $v(0,t) = 0$.

We will look for a function $q^*(t)$ so that a type t agent purchases $q^*(t)$. Any candidate function $q(t)$ must satisfy

(IC) $v(q(s),t)-R(q(s)) \leq v(q(t),t) - R(q(t)) = \pi(t)$

yielding the first order condition

$$v_q(q(t),t) - R'(q(t)) = 0.$$

and thus (envelope theorem)

$$\pi'(t) = v_t(q(t),t).$$

As before, the individual rationality constraint requires

(IR) $\pi(t) \geq 0$.

However, since π is nondecreasing (as $v_t(q,t) = \int_0^q v_{qt}(x,t)dx = \int_0^q p_t(x,t)dx \geq 0$), IR is equivalent to $\pi(0)\geq 0$.

Therefore,

$$\int_0^1 \pi(t) f(t)dt = -\pi(t)(1-F(t)) \Big|_0^1 + \int_0^l \pi'(t)(1-F(t))dt$$

$$-\pi(0) + \int_0^1 v(q(t),t)(1-F(t))dt$$

Consequently, the monopolist's profit can be expressed as:

$$\int_0^1 R(q(t)) - cq(t)) f(t)dt = \int_0^1 (v(q(t),t) - \pi(t) - cq(t)) f(t)dt$$

$$= -\pi(0) + \int_0^1 \left(v(q(t),t) - \frac{1-F(t)}{f(t)} v_t(q(t),t) - cq(t) \right) f(t)dt.$$

Because of IR, the monopolist will set $\pi(0)=0$; otherwise he charge all buyers the additional amount $\pi(0)$, increasing his profit and still satisfying IR and IC. Maximizing pointwise gives:

(2) $$p(q^*(t),t) - \frac{1-F(t)}{f(t)} p_t(q^*(t),t) - c = 0.$$

Necessity and sufficiency: The IC constraint holds if and only if the first order condition for the buyer's maximization holds, and q is nondecreasing.

Let $u(s,t)=v(q(s),t) - R(q(s))$, which is what a type t agent gets if he buys the quantity slated for type s. Then IC can be written

$$u(s,t) \leq u(t,t).$$

Denote partial derivatives with subscripts. Necessarily, $u_1(t,t)=0$ and $u_{11}(t,t)\leq 0$. Totally differentiating the first gives $u_{11}(t,t) + u_{12}(t,t)=0$, so the second order condition can be rewritten $u_{12}(t,t)\geq 0$. Therefore, necessarily,

$$0 \leq v_{qt}(q(t),t)\, q'(t),$$

which forces q nondecreasing, since $v_{qt}=p_t>0$. Now turn to sufficiency. Note that, if q is nondecreasing, then u_{12} is everywhere nonnegative. Thus, for $s<t$, $u_1(s,t) \leq u_1(s,s) = 0$, and for $s>t$,

$u_1(s,t) \geq u_1(s,s) = 0$. Thus, u is increasing in s for $s<t$, and decreasing in s for $s>t$, and therefore u is maximized at $s=t$, and IC holds.

Thus, $q^{*\prime}(t) \geq 0$ is both necessary and sufficient for the solution to

$$R'(q^*(t)) = p(q^*(t),t)$$

$$R(q^*(0)) = v(q^*(0),0)$$

to maximize the monopolist's profit, where q^* is given by (2). This defines the optimal R.

Observations:

(1) The highest type consumer gets the efficient quantity, in that price $p(q^*(1),1) = c$, marginal cost

(2) Those with greater demand (high t's, since $p_t>0$) obtain at least as much of the good, and sometimes more, than those with lower demand.

(3) All agents except the highest type get less than the efficient quantity

This follows from $p(q^*(t),t) - c = \frac{1-F(t)}{f(t)} p_t(q^*(t),t) > 0$,

since

$$v_t(q,t) = \int_0^q v_{qt}(r,t)dr > 0.$$

(4) If the optimal quantity is decreasing in some neighborhood, then a flat spot results from the optimization and an interval of types are treated equally (called *pooling*).

(5) The monopolist's solution may be implemented using a nonlinear price schedule. Under some circumstances, it may be implemented using a menu of linear price schedules, that is, offering lower marginal costs, at a higher fixed cost, much like phone companies do.

(6) The solution can be interpreted according to the elasticity formula already given. Let $y = 1 - F(t)$ represent the number of consumers willing to buy $q(t)$ at price $p(q(t),t)$. Note that

$$\frac{p(q(t),t)-c}{p(q(t),t)} = \frac{1-F(t)}{f(t)} \frac{p_t(q(t),t)}{p(q(t),t)} = \frac{\frac{\partial}{\partial t}\log(p(q,t))}{-\frac{\partial}{\partial t}\log(1-F(t))} = -\frac{1}{\frac{p}{y}\frac{dy}{dp}}.$$

Quality Premia

We can actually cover this case without doing any work, by merely reinterpreting the quantity discounts model. Suppose the monopolist faces two types of consumers, L and H. The monopolist has at his disposal a range of qualities to offer. Both types value higher quality more, but the H type values an increase in quality more than the L type, that is, $v_L'(q) < v_H'(q)$. In this case, the monopolist will offer two qualities, one high and one low. The high quality good will be efficient, i.e. sets the marginal value of quality to the marginal cost. The low quality, however, will be worse than efficient. That is, the monopolist will intentionally make the low quality good worse, so as to be able to charge more for the high quality good.

The case where c=0 is especially interesting, because this is the case in which quality is free, say, up to an upper bound $\bar{q}$. One can imagine that the monopolist only produces one good, and, at no cost, can make it lower quality, say, by hitting it with a hammer. In this case, the monopolist will *still* offer two qualities, that is, the monopolist will intentionally damage a portion of the goods he sells, so as to be able to segment the market. It is worth thinking about whether this is what goes on at outlet malls and stores like *Sam's* and *The Price Club*. Manufacturers create inconvenient sizes of products, or locate outlets at distant (although not necessary less expensive) locations, in order to be able to charge less to the more price sensitive segment of their market.

It is a straightforward exercise to adapt the two type model so that it is *more* costly to offer lower quality, that is, the manufacturer has to take an existing product and pay to have it damaged. The only thing that is needed is to replace the cost $c(q_L+q_H)$ with $c_L(q_L) + c_H(q_H)$, where $c_L > c_H$. In this case, the manufacturer may still offer the low quality, that is, pay extra to have some of the goods damaged. The objective of this action is the same, that is, to deter the high demand types from buying the low quality, by reducing the low quality below efficient levels.

Tie-ins

"Buy a suit and get an electric drill." -Detroit TV Ad, 1981

"Shoe: Buy one, get one free". -South Carolina Billboard, 1987

Tie-ins arise whenever a manufacturer requires the purchase of one product in order to purchase another product. Thus, if an automobile manufacturer required you to use their parts when you had the car serviced, a tie-in would have occurred.

Reasons for Tie-ins

1. Lower Cost

a. save on packaging, e.g. right and left shoes not sold separately. In particular, I bundled my graduate and undergraduate notes to save on writing two separate sets of notes…
b. save on sorting, e.g. bags of oranges of average quality, de Beers's bags of diamonds.

2. Evade price controls

Australia tried to hold down the price of new cars, so the price of used cars went up. Australia then regulated the price of used cars, and the price of car radios, tied to the sale of a used car, went up.

3. Circumvent other regulation

Used as justification in A.T.&T. breakup.

4. Offer Secret Price Cuts

Firms in a cartel may want to lower price without their competitors knowing.

5. Assure Quality

Kodak once sold film with development included, which rules out consumers mistakenly blaming Kodak for bad local film development.

6. Price Discriminate

There are two ways one can tie sales. One is bundling (such as selling new cars with tires already included). Bundling is legal. The second way is a requirements condition [if you use my computer, you must use my computer cards], which is illegal. The basic distinction is as follows. Once a product has been purchased, the manufacturer loses control, and can't force a consumer to do anything with it. For example, a manufacturer can't condition a warranty on the use of the manufacturer's parts if other parts of equal quality are available. The manufacturer can condition the warranty on the use of parts of adequate quality. If your car engine blows up because you put a bad water pump from another manufacturer on it, then this can void the warranty. If you can show, however, that the water pump was of equal or better quality than the original car manufacturer's pump, then your warranty will still be valid.

Many of the lawsuits concerning tie-ins are between franchisees and franchisors, and concern whether a franchisee has to by the franchisor's products (e.g. does a McDonald's franchise have to buy the napkins and coffee stirrers sold by the McDonald's corporation?). This occurs because of the main things a franchise offers is a nationwide quality standard. Thus, people go into McDonald's when travelling in California because they are familiar with the quality in Texas, and expect that quality to be the same. An individual franchise, however, often has an incentive to cut quality (e.g. filthy bathrooms) because it saves on costs, and most of the impact is felt by other outlets (e.g. a McDonald's on an interstate highway gets little repeat business, so it loses few customers because of low quality, but people who do go there are less likely to go into other McDonalds. For this reason, McDonald's polices the quality very carefully). This is only half of the story. The franchisor has an incentive, once the business is not growing rapidly, to try to increase the prices of inputs to the franchisees, who have sunk a large investment in the business and are unlikely to go bankrupt because of an input price increase. Thus, once the business has become mature, a franchisor that requires the use of its own inputs might raise the price to the franchisees. So the lawsuits usually revolve on (i) is the franchisee trying to cut quality by using cheaper inputs than the franchisor's own brand, or (ii) is the franchisor overcharging for the inputs?

In most of these cases, experts are brought in to compare the quality of the franchisor's product and the competing brand that the franchise wants to buy.

Generally, a franchisor has the right to insist on a minimum quality, but does not have the right to insist on the use of its own products, if products of equal quality are available.

Bundling, on the other hand, is generally legal. However, offering a lower price for two products together than the sum of the individual prices, called *mixed bundling* (*pure bundling* is when you don't offer the two products separately as well as in a bundle, e.g. they don't sell new cars without tires, so cars + tires are pure bundling, but they do sell new cars without radios, new cars with radios, and radios, so cars + radios are mixed bundling), can be illegal if it is found to be price discrimination (see Robinson-Patman Act).

It turns out that, under reasonable specifications of preferences, a monopolist *always* prefers mixed bundling to no bundling, that is, the monopolist will always set a price for the bundle lower than the sum of the individual prices. See McAfee, McMillan, and Whinston, ***Quarterly Journal of Economics***, 1989.